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Subject – Vector Calculus

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Chapter 4: Vector-Valued Functions

Topic- Curves in Space, Limits and Continuity, Derivatives and Motion, Differentiation Rules for Vector Function, Vector Functions of Constant Length

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Maths – II (MT – 242 (A))

Vector calculus

Chapter – I

Vector – Valued Functions

Def^n – Vector valued function –

A vector valued function is defined as

$$\bar{r}(t) = f(t)\,\bar{i} + g(t)\,\bar{j}$$

or $$\bar{r}(t) = f(t)\,\bar{i} + g(t)\,\bar{j} + h(t)\,\bar{k}$$

where the components f, g & h are real valued function of the parameter t.

The vector valued f^n are also written in the following form

$$\bar{r}(t) = (f(t), g(t))$$

or

$$\bar{r}(t) = (f(t), g(t), h(t))$$

Ex. Find the domain of $\bar{r}(t)$ & the value of $\bar{r}(t_0)$

1) $\bar{r}(t) = \cos t\,\bar{i} - 3t\,\bar{j}$, $t_0 = \pi$

Domain $= \{t \in \mathbb{R}\} = \mathbb{R}$

$\cos n\pi = (-1)^n$

$\sin n\pi = 0$

$$\bar{r}(\pi) = \cos\pi\,\bar{i} - 3\pi\,\bar{j}$$
$$= (-1)\,\bar{i} - 3\pi\,\bar{j}$$
$$= -\bar{i} - 3\pi\,\bar{j}$$

2) $\bar{r}(t) = \sqrt{3t+1}\,\bar{i} + t^2\,\bar{j}$, $t_0 = 1$

domain $= \{t \in \mathbb{R} \,/\, 3t+1 \geq 0\}$

$= \{t \in \mathbb{R} \,/\, t \geq -\frac{1}{3}\}$

$\sqrt{4} = \pm 2$

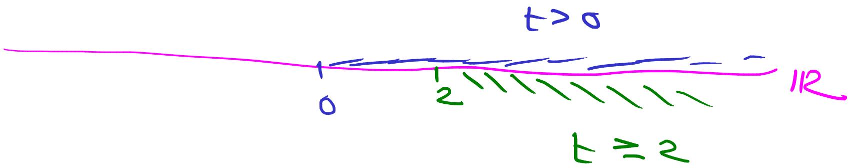
$$\bar{r}(1) = \sqrt{3(1)+1}\,\bar{i} + 1\,\bar{j}$$
$$= 2\,\bar{i} + \bar{j} \quad \text{or} \quad \pm 2\,\bar{i} + \bar{j}$$

3) $\bar{r}(t) = \cos\pi t\,\bar{i} - \ln t\,\bar{j} + \sqrt{t-2}\,\bar{k}$, $t_0 = 3$

$\Rightarrow$ $\cos\pi t \Rightarrow t \in \mathbb{R}$

$\ln t \Rightarrow t > 0$

$\sqrt{t-2} \Rightarrow t-2 \geq 0 \Rightarrow t \geq 2$



domain $= \{t \in \mathbb{R} \mid t \geq 2\}$

$\bar{r}(3) = \cos 3\pi\,\bar{i} - \ln 3\,\bar{j} + \sqrt{3-2}\,\bar{k}$

$= (-1)^3\,\bar{i} - \ln 3\,\bar{j} + 1\,\bar{k}$

$= -\bar{i} - \ln 3\,\bar{j} + \bar{k}$

Ex. $\bar{r}(t) = 2e^{-t}\,\bar{i} + \sin^{-1} t\,\bar{j} + \ln(1-t)\,\bar{k}$, $t_0 = 0$

$e^{-t} \Rightarrow t \in \mathbb{R}$

$\sin^{-1}(t) \Rightarrow -1 \leq t \leq 1$

$\ln(1-t) \Rightarrow 1-t > 0$

i.e.

$1 > t$

or $t < 1$

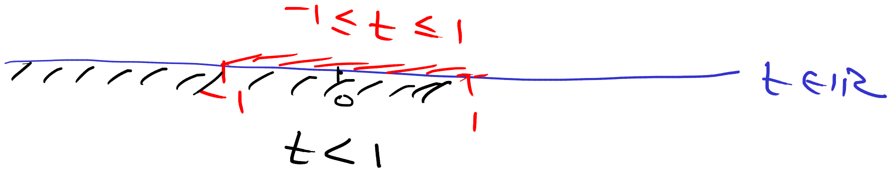
$\theta = \sin^{-1}(2)$

$\Rightarrow \sin\theta = 2$

$-1 \leq \sin\theta \leq 1$

for $\cos^{-1}(t)$

$-1 \leq t \leq 1$



$-1 \leq t < 1$

domain of $\bar{r}(t) = \{t \in \mathbb{R} \mid -1 \leq t < 1\}$

Ex. $\bar{r}(t) = 3\tan t\,\bar{i} + 4\sec t\,\bar{j} + 5t\,\bar{k}$, $\bar{r}(0)$, $\bar{r}(\frac{\pi}{2})$

$\bar{r}(\frac{2\pi}{3})$

$\Rightarrow$ $\tan t = \frac{\sin t}{\cos t}$, $\tan t$ is not defined if

$\cos t = 0 \Rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{4\pi}{2}$

$\cos t = 0 \Rightarrow t = (2n+1)\frac{\pi}{2}, \quad n = 0, \pm 1, \pm 2, \ldots$

$\& \ \sin t = 0 \Rightarrow t = n\pi$

$\tan t$ exist if $t \neq (2n+1)\frac{\pi}{2}$

$\sec t = \frac{1}{\cos t}$ exist if $t \neq (2n+1)\frac{\pi}{2}$

$\therefore$ domain $\bar{r}(t) = \{t \in \mathbb{R} \,/\, t \neq (2n+1)\frac{\pi}{2}\}$

$$\bar{r}(0) = 3\tan(0)\bar{i} + 4\sec(0)\bar{j} + 5(0)\bar{k}$$
$$= 0\bar{i} + 4(1)\bar{j} + 0\bar{k} = 4\bar{j}$$

$$\bar{r}\left(\frac{\pi}{2}\right) = 3\tan\left(\frac{\pi}{2}\right)\bar{i} + 4\sec\left(\frac{\pi}{2}\right)\bar{j} + 5\frac{\pi}{2}\bar{k}$$
$$= \text{does not exist.}$$

$\tan\frac{\pi}{2} = \frac{\sin\frac{\pi}{2}}{\cos\pi/2} = \frac{1}{0}$ = not define.

$$\bar{r}\left(\frac{2\pi}{3}\right) = 3\tan\left(\frac{2\pi}{3}\right)\bar{i} + 4\sec\left(\frac{2\pi}{3}\right)\bar{j} + 5\left(\frac{2\pi}{3}\right)\bar{k}$$
$$= 3(-\sqrt{3})\bar{i} + 4(-2)\bar{j} + \frac{10\pi}{3}\bar{k}$$
$$= -3\sqrt{3}\,\bar{i} - 8\bar{j} + \frac{10\pi}{3}\bar{k}.$$

* **Limit of vector valued Function** –

Let $\bar{r}(t) = f(t)\bar{i} + g(t)\bar{j} + h(t)\bar{k}$ be a vector function with domain D & let $\bar{L} = l_1\bar{i} + l_2\bar{j} + l_3\bar{k}$ be a vector. Then we say that $\bar{L}$ is the limit of the vector-valued function $\bar{r}(t)$ as $t \to t_0$ if for every $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $\forall t \in D$

$$|\bar{r}(t) - \bar{L}| < \epsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta$$

we can denote this as

$$\lim_{t \to t_0} \bar{r}(t) = \bar{L}$$

Remark – If $\bar{r}(t) = f(t)\bar{i} + g(t)\bar{j} + h(t)\bar{k}$ and $\bar{L} = l_1\bar{i} + l_2\bar{j} + l_3\bar{k}$ then $\lim_{t \to t_0} \bar{r}(t) = \bar{L}$ if & only if $\lim_{t \to t_0} f(t) = l_1$, $\lim_{t \to t_0} g(t) = l_2$

& $\lim_{t\to t_0} h(t) = l_3$, Hence we have

$$\lim_{t\to t_0} \bar{r}(t) = \left[\lim_{t\to t_0} f(t)\right]\bar{i} + \left[\lim_{t\to t_0} g(t)\right]\bar{j} + \left[\lim_{t\to t_0} h(t)\right]\bar{k}$$

* **Continuity of vector valued function-**

Let $\bar{r}(t) = f(t)\bar{i} + g(t)\bar{j} + h(t)\bar{k}$ be a vector function with domain D. Then the vector-valued function $\bar{r}(t)$ is said to be continuous at $t = t_0$ if

$$\lim_{t\to t_0} \bar{r}(t) = \bar{r}(t_0)$$

The function is continuous if it is continuous at every point in its domain.

* **Derivative of vector-valued function-**

Let $\bar{r}(t)$ be a vector-valued function with domain D. Then the vector valued function $\bar{r}(t)$ is said to be differentiable at $t = t_0$ if

$$\lim_{\Delta t\to 0} \frac{\bar{r}(t_0+\Delta t) - \bar{r}(t_0)}{\Delta t}$$ is exist and finite.

It is denoted by

$$\frac{d}{dt}\bar{r}(t_0) \quad \text{or} \quad \bar{r}'(t_0) \quad \text{or} \quad \left(\frac{d\bar{r}}{dt}\right)_{t=t_0}$$

$$\frac{d}{dr}\bar{r}(t) = \bar{r}'(t) = \lim_{\Delta t\to 0} \frac{\bar{r}(t+\Delta t) - \bar{r}(t)}{\Delta t}$$

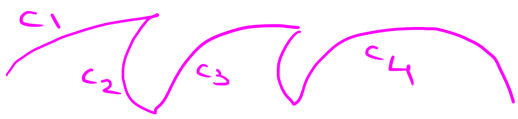
* **Remark-**

1] A vector function $\bar{r}(t)$ is differentiable if it is differentiable at every point of its domain.

2] A vector function $\bar{r}(t)$ is diff. function of t iff the components $f(t), g(t), h(t)$ are diff. functions of t.

3] The curve trace by $\bar{r}(t)$ is smooth if $\frac{d\bar{r}}{dt}$ is continuous & $\frac{d\bar{r}}{dt} \neq 0$ for any t that is if f, g & h have conti first derivatives that are not simultaneously zero.

4) A curve that is made up of a finite number of smooth curves pieced together in a continuous fashion is called piecewise smooth.



5) If $\bar{r}$ is position vector of a particle moving along a smooth curve in space then

i) $\bar{v}(t) = \frac{d}{dt}\bar{r}(t)$ is the particle velocity vector, tangent to the curve at any time t, the direction of $\bar{v}$ is the direction of motion.

ii) speed is the magnitude of velocity $\bar{v}(t)$

i.e. speed $= |\bar{v}(t)|$

If $\bar{r} = a\bar{i} + b\bar{j} + c\bar{k}$

$|\bar{r}| = \sqrt{a^2+b^2+c^2}$

iii) Acceleration is the derivative of velocity i.e.

$$\bar{a}(t) = \frac{d}{dt}\bar{v}(t) = \frac{d^2}{dt^2}\bar{r}(t)$$

iv) The unit vector $\hat{v} = \frac{\bar{v}}{|\bar{v}|}$ is the direction of motion at time t.

v) The tangent line to a smooth curve $\bar{r}(t) = f(t)\bar{i} + g(t)\bar{j} + h(t)\bar{k}$ at $t = t_0$ is the line that passes through the point $(f(t_0), g(t_0), h(t_0))$ parallel to $\bar{v}(t_0)$, the curves velocity vector at t_0.

[Eqn of line passing through (x_0, y_0, z_0) and d.r.s. a, b, c is

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} = \lambda$$

$$\frac{x-x_0}{a} = \lambda \Rightarrow x = a\lambda + x_0$$

$$y = y_0 + b\lambda \quad \& \quad z = z_0 + c\lambda$$

Ex. Evaluate

$$\lim_{t \to 0} [e^{-t} \cos t \, \bar{i} + e^{-t} \sin t \, \bar{j} + e^{-t} \bar{k}]$$

$$= e^{0} \cos 0 \, \bar{i} + e^{0} \sin 0 \, \bar{j} + e^{0} \bar{k}$$

$$= \bar{i} + 0\bar{j} + \bar{k} \quad = \bar{i} + \bar{k}$$

Ex. Evaluate

$e^{-\infty} = 0$

$e^{\infty} = \infty$

$$\lim_{t \to \infty} [2e^{-t} \bar{i} + e^{-t} \bar{j} + e^{-2t} \bar{k}]$$

$$= 2e^{-\infty}\bar{i} + e^{-\infty} \bar{j} + e^{-\infty} \bar{k} = \bar{0}$$

Ex. $$\lim_{t \to 0} \left[\frac{\sin t}{t} \bar{i} + \frac{\tan^2 2t}{\sin 2t} \bar{j} + \frac{t^3 - 8}{t+2} \bar{k} \right]$$

$$= \lim_{t \to 0} \frac{\sin t}{t} \bar{i} + \lim_{t \to 0} \frac{\tan^2 2t}{\sin 2t} \bar{j}$$

$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

$$+ \lim_{t \to 0} \frac{t^3 - 8}{t+2} \bar{k}$$

$\lim_{t \to 0} \frac{\tan^2 0}{\sin 0} = \frac{0}{0}$

L'Hospital rule

$\lim_{t \to 0} \frac{2\tan 2t \cdot \sec^2 2t \cdot \cancel{2}}{\cos 2t \cdot \cancel{2}}$

$$= \bar{i} + \lim_{t \to 0} \frac{2\tan 2t \cdot \sec^2 2t \cdot \cancel{2}}{\cos 2t \cdot \cancel{2}} \bar{j}$$

$$+ \frac{-8}{2} \bar{k}$$

$\tan 0 = 0$

$$= \bar{i} + \frac{2\tan 0 \cdot \sec^2 0}{\cos 0} \bar{j} - 4\bar{k}$$

$$= \bar{i} + 0\bar{j} - 4\bar{k} = \bar{i} - 4\bar{k}$$

Ex. $$\lim_{t \to \frac{\pi}{6}} [\cos^2 t \, \bar{i} + \sin^2 t \, \bar{j} + \bar{k}]$$

$\cos \frac{\pi}{6} = \cos 30$

$= \frac{\sqrt{3}}{2}$

$\sin 30 = \frac{1}{2}$

$$= \cos^2 \frac{\pi}{6} \bar{i} + \sin^2 \frac{\pi}{6} \bar{j} + \bar{k}$$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 \bar{i} + \left(\frac{1}{2}\right)^2 \bar{j} + \bar{k}$$

$$= \frac{3}{4} \bar{i} + \frac{1}{4} \bar{j} + \bar{k}$$

Ex. $$\lim_{t \to \infty} \left[2e^{-2t} \bar{i} + \frac{2t+3}{3t-1} \bar{j} + \tan^{-1} 2t \, \bar{k}\right]$$

$$= \lim_{t\to\infty} 2e^{-2t}\,\bar{i} + \lim_{t\to\infty} \frac{2t+3}{3t-1}\,\bar{j} + \lim_{t\to\infty} \tan^{-1}2t\,\bar{k}$$

L' Hospital rule

$$= 2e^{-\infty}\,\bar{i} + \lim_{t\to\infty} \frac{2}{3}\,\bar{j} + \tan^{-1}\infty\,\bar{k}$$

$$= 2(0)\,\bar{i} + \frac{2}{3}\,\bar{j} + \frac{\pi}{2}\,\bar{k}$$

$\sin(t-2)^2$
$= \sin^2(t-2)$

$$= \frac{2}{3}\,\bar{j} + \frac{\pi}{2}\,\bar{k}$$

Ex. $$\lim_{t\to 2}\left[\frac{\sin^2(t-2)}{t-2}\,i + \frac{t^2-4}{t-2}\,j\right]$$

L' Hospital rule

$$= \lim_{t\to 2} \frac{2\sin(t-2)\cdot\cos(t-2)}{1}\,\bar{i} + \lim_{t\to 2}\frac{2t}{1}\,\bar{j}$$

$$= 2\sin 0\cos 0\,\bar{i} + 4\bar{j} = 4\bar{j}$$

Ex. Discuss the conti of the following fns

$$\bar{f}(t) = \begin{cases} \sin\frac{t}{2}\,\bar{i} + \cos\frac{2t}{3}\,\bar{j} + \tan\frac{5t}{4}\,\bar{k}, & t \neq \pi \\ \bar{i} - \frac{1}{2}\bar{j} + \bar{k}, & t = \pi \end{cases}$$

$\Rightarrow$ consider

$$\lim_{t\to\pi} \bar{f}(t) = \lim_{t\to\pi}\left[\sin\frac{t}{2}\,\bar{i} + \cos\frac{2t}{3}\,\bar{j} + \tan\frac{5t}{4}\,\bar{k}\right]$$

$$= \sin\frac{\pi}{2}\,\bar{i} + \cos\frac{2\pi}{3}\,\bar{j} + \tan\frac{5\pi}{4}\,\bar{k}$$

$$= 1\,\bar{i} - \frac{1}{2}\bar{j} + 1\,\bar{k}$$

$$= \bar{i} - \frac{1}{2}\bar{j} + \bar{k}$$

$\cos\frac{2\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right)$
$= -\cos\frac{\pi}{3}$
$= -\frac{1}{2}$

at $t = \pi$

$$\bar{f}(t) = \bar{f}(\pi) = \bar{i} - \frac{1}{2}\bar{j} + \bar{k}$$

$\therefore \lim_{t\to\pi} \bar{f}(t) = \bar{f}(\pi)$

$\tan\frac{5\pi}{4} = \tan\left(\pi + \frac{\pi}{4}\right)$
$= \tan\frac{\pi}{4}$
$= 1$

$\therefore \bar{f}(t)$ is conti at $t = \pi$.

Ex.

$$\bar{f}(t) = \begin{cases} t^3\,\bar{i} + \sin\frac{\pi}{2}t\,\bar{j} + \ln(t+2)\,\bar{k} & , t \neq -1 \\ \bar{i} - \bar{j} & \text{if } t = -1 \end{cases}$$

$$\Rightarrow \lim_{t \to -1} \bar{f}(t) = \lim_{t \to -1} \left[t^3\,\bar{i} + \sin\frac{\pi}{2}t\,\bar{j} + \ln(t+2)\,\bar{k} \right]$$

$$= -\bar{i} + \sin\left(-\frac{\pi}{2}\right)\bar{j} + \ln(-1+2)\,\bar{k}$$

$$= -\bar{i} - \bar{j} + 0$$

$\sin(-\theta) = -\sin\theta$

$$= -\bar{i} - \bar{j}$$

$\therefore \lim_{t \to -1} \bar{f}(t) \neq \bar{f}(-1)$

$\therefore \bar{f}(t)$ is disconti at $t = -1$

Ex.

$$\bar{f}(t) = \begin{cases} \dfrac{t^2-1}{\ln t}\,\bar{i} + \dfrac{\sqrt{t}-1}{1-t}\,\bar{j} + \tan^{-1}(t)\,\bar{k} & , t \neq 1 \\ 2\bar{i} - \frac{1}{2}\bar{j} + \frac{\pi}{4}\bar{k} & , t = 1 \end{cases}$$

$$\Rightarrow \lim_{t \to 1} \bar{f}(t) = \lim_{t \to 1} \left[\frac{t^2-1}{\ln t}\right]\bar{i} + \lim_{t \to 1} \frac{\sqrt{t}-1}{1-t}\,\bar{j} + \lim_{t \to 1} \tan^{-1}(t)\,\bar{k}$$

by L'Hospital rule

$$= \lim_{t \to 1} \frac{2t}{\frac{1}{t}}\,\bar{i} + \lim_{t \to 1} \frac{\frac{1}{2\sqrt{t}}}{-1}\,\bar{j} + \lim_{t \to 1} \tan^{-1}(1)\,\bar{k}$$

$$= 2\bar{i} - \frac{1}{2}\bar{j} + \frac{\pi}{4}\bar{k}$$

since $\bar{f}(1) = 2\bar{i} - \frac{1}{2}\bar{j} + \frac{\pi}{4}\bar{k}$

$\therefore \lim_{t \to 1} \bar{f}(t) = \bar{f}(1)$

$\therefore \bar{f}(t)$ is conti at $t = 1$

Ex. $\bar{F}(t) = \begin{cases} \frac{\sin t}{t}\,\bar{i} + \frac{\tan^2 t}{\sin 2t}\,\bar{j} + \frac{t^3-8}{t+2}\,\bar{k}, & t \neq 0 \\ \bar{i} + \bar{j} - 4\bar{k}, & t = 0 \end{cases}$

$$\Rightarrow \lim_{t\to 0} \bar{F}(t) = \lim_{t\to 0} \frac{\sin t}{t}\,\bar{i} + \lim_{t\to 0} \frac{\tan^2 t}{\sin 2t}\,\bar{j} + \lim_{t\to 0} \frac{t^3-8}{t+2}\,\bar{k}$$

$$= 1\bar{i} + \lim_{t\to 0} \frac{2\tan t \cdot \sec^2 t}{\cos 2t \cdot 2}\,\bar{j} + \frac{(-8)}{2}\,\bar{k}$$

$$= \bar{i} + 0\bar{j} - 4\bar{k}$$

$$= \bar{i} - 4\bar{k}$$

Now $\bar{F}(0) = \bar{i} + \bar{j} - 4\bar{k}$

$\therefore \lim_{t\to 0} \bar{F}(t) \neq \bar{F}(0)$

$\therefore \bar{F}(t)$ is not conti at $t = 0$.

*** Chain Rule –**

Let $\bar{U}(s) = a(s)\bar{i} + b(s)\bar{j} + c(s)\bar{k}$ be a vector function of s and $s = f(t)$ is a differential scalar function of t then

$$\frac{d}{dt}\bar{U}(s) = \frac{d}{ds}\bar{U}(s) \cdot \frac{ds}{dt}$$

$$= \frac{d}{ds}\bar{U}(f(t)) \cdot \frac{d}{dt} f(t)$$

*** Thm – Vector functions of constant length –**

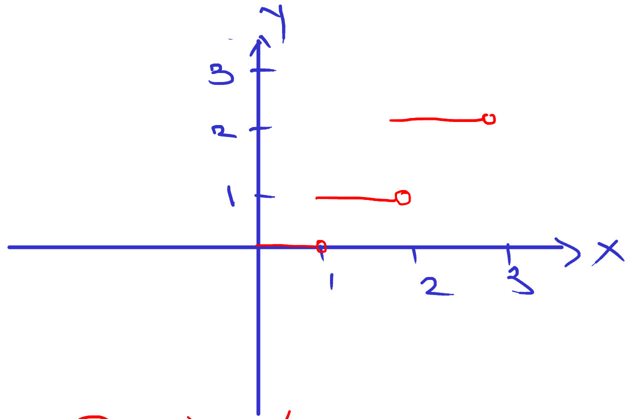
If $\bar{r}(t)$ is a differentiable vector function of t and length of $\bar{r}(t)$ is constant then

$$\bar{r} \cdot \frac{d\bar{r}}{dt} = 0$$

Ex. Discuss the continuity of the function

$$\bar{F}(t) = \cos t\,\bar{i} + \sin t\,\bar{j} + [t]\,\bar{k}$$

where $[t]$ is the greatest integer function.



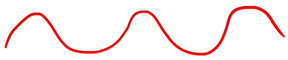
$f(x) = n \quad , \quad n \le x < n+1$

$f(2.1) = 2 \qquad 2 < 2.1 < 3$

$f(4.5) = 4 \qquad 4 < 4.5 < 5$

$f(x) = 5 \qquad 5 \le x < 6$

x	$y = f(x)$
$0 \le x < 1$	0
$1 \le x < 2$	1



$\sin x / \cos x$

$\sin t$ & $\cos t$ are conti for $\forall t \in \mathbb{R}$. but $[t]$ is disconti at every integers. Therfore $f(t)$ is disconti at every integers.

Ex. If $\bar{r}(t) = \frac{t}{t+1}\bar{i} + \frac{1}{t}\bar{j}$ is the position value of particle in the xy plane at time t. Find an equation in x & y whose graph is the path of the particle. Then find the particles velocity & acceleration vectors at $t = -\frac{1}{2}$.

$\Rightarrow$ since

$$\bar{r} = x\bar{i} + y\bar{j}$$

$\bar{r} = x\bar{i} + y\bar{j}$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\therefore \quad x = \frac{t}{t+1} \quad \& \quad y = \frac{1}{t}$$

$$\frac{1}{x} = \frac{t+1}{t} = 1 + \frac{1}{t} = 1 + y$$

$$\therefore \quad \frac{1}{x} - 1 = y$$

$$\text{or} \quad y = \frac{1}{x} - 1$$

This is required eqn whose graph is path of the particle.

$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$

$$\bar{r}(t) = \frac{t}{t+1}\bar{i} + \frac{1}{t}\bar{j}$$

$$\text{velocity} = \bar{v}(t) = \frac{d}{dt}\bar{r} = \frac{(t+1)(1) - t(1)}{(t+1)^2}\bar{i} - \frac{1}{t^2}\bar{j}$$

$$\bar{v}(t) = \frac{1}{(t+1)^2}\bar{i} - \frac{1}{t^2}\bar{j}$$

at $t = -\frac{1}{2}$

$$\bar{v}\left(-\tfrac{1}{2}\right) = \frac{1}{\frac{1}{4}}\bar{i} - \frac{1}{\frac{1}{4}}\bar{j} = 4\bar{i} - 4\bar{j}$$

$$\text{acc.} = \bar{a}(t) = \frac{d}{dt}\bar{v}(t) = \frac{-2}{(t+1)^3}\bar{i} + \frac{2}{t^3}\bar{j}$$

at $t = -\frac{1}{2}$

$$\bar{a}\left(-\tfrac{1}{2}\right) = \frac{-2}{\frac{1}{8}}\bar{i} + \frac{2}{-\frac{1}{8}}\bar{j}$$

$$= -16\bar{i} - 16\bar{j}$$

Ex. If $\bar{r}(t) = \sec t\,\bar{i} + (\tan t)\bar{j} + \frac{4}{3}t\,\bar{k}$ is the position of the particle in the space at time t. Find the particle velocity and acceleration vectors at $t = \frac{\pi}{6}$. Write the particle's velocity at that time as the product of its speed & direction.

$\Rightarrow$ $\bar{r}(t) = \sec t\,\bar{i} + \tan t\,\bar{j} + \frac{4}{3}t\,\bar{k}$

$$\text{velocity} = \bar{v}(t) = \frac{d\bar{r}}{dt} = \sec t \cdot \tan t\,\bar{i} + \sec^2 t\,\bar{j} + \frac{4}{3}\bar{k}$$

$$\text{accl}^n = \bar{a}(t) = \frac{d\bar{v}}{dt} = [\sec t \cdot \sec^2 t + \sec t \tan t \cdot \tan t]\bar{i} + 2\sec t \cdot \sec t \cdot \tan t\,\bar{j} + 0\bar{k}$$

at $t = \frac{\pi}{6}$

$$\bar{v}\left(\tfrac{\pi}{6}\right) = \sec\tfrac{\pi}{6}\tan\tfrac{\pi}{6}\,\bar{i} + \sec^2\tfrac{\pi}{6}\,\bar{j} + \tfrac{4}{3}\bar{k}$$

$$= \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}\bar{i} + \frac{4}{3}\bar{j} + \frac{4}{3}\bar{k}$$

$$= \frac{2}{3}\bar{i} + \frac{4}{3}\bar{j} + \frac{4}{3}\bar{k}$$

$$\sec\tfrac{\pi}{6} = \frac{1}{\cos \pi/6} = \frac{1}{\cos 30} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

$$\tan 30 = \frac{\sin 30}{\cos 30} = \frac{1/2}{\sqrt{3}/2}$$

at $t = \frac{\pi}{6}$

$$\bar{a}\left(\tfrac{\pi}{6}\right) = \left[\frac{8}{3\sqrt{3}} + \frac{2}{3\sqrt{3}}\right]\bar{i} + \frac{2}{3\sqrt{3}}\bar{j} + 0\bar{k}$$

$$= \frac{10}{3\sqrt{3}}\bar{i} + \frac{2}{3\sqrt{3}}\bar{j}$$

$$\bar{r} = a\bar{i} + b\bar{j} + c\bar{k}$$

$$|\bar{r}| = \sqrt{a^2+b^2+c^2}$$

$$\text{speed} = \left|\bar{v}\left(\tfrac{\pi}{6}\right)\right| = \sqrt{\left(\tfrac{2}{3}\right)^2 + \left(\tfrac{4}{3}\right)^2 + \left(\tfrac{4}{3}\right)^2}$$

$$= \sqrt{\frac{4}{9} + \frac{16}{9} + \frac{16}{9}} = \frac{6}{3}$$

$$\text{direction} = \hat{v}\left(\frac{\pi}{6}\right) = \frac{\bar{v}}{|\bar{v}|} = \frac{1}{\frac{6}{3}}\left[\frac{2}{3}\bar{i} + \frac{4}{3}\bar{j} + \frac{4}{3}\bar{k}\right]$$

$$= \frac{1}{3}\bar{i} + \frac{2}{3}\bar{j} + \frac{2}{3}\bar{k}$$

$$\text{velocity} = (\text{speed})(\text{direction})$$

$$= \frac{6}{3}\left[\frac{1}{3}\bar{i} + \frac{2}{3}\bar{j} + \frac{2}{3}\bar{k}\right]$$

Ex. If $\bar{r}(t) = e^{-t}\bar{i} + (2\cos 3t)\bar{j} + 2\sin 3t\,\bar{k}$ is the position of the particle in the space at time t. Find the particle velocity & acc. vectors at $t=0$. Write the particle velocity at that time as the product of its speed and direction.

$\Rightarrow$

$$\bar{r}(t) = e^{-t}\bar{i} + (2\cos 3t)\bar{j} + 2\sin 3t\,\bar{k}$$

$$\bar{v}(t) = \frac{d\bar{r}}{dt} = -e^{-t}\bar{i} - 6\sin 3t\,\bar{j} + 6\cos 3t\,\bar{k}$$

$$\bar{a}(t) = \frac{d\bar{v}}{dt} = e^{-t}\bar{i} - 18\cos 3t\,\bar{j} - 18\sin 3t\,\bar{k}$$

at $t=0$

$$\bar{v}(0) = -\bar{i} - 0\bar{j} + 6\bar{k} = -\bar{i} + 6\bar{k}$$

$$\text{speed} = |\bar{v}(0)| = \sqrt{(-1)^2 + 0^2 + 6^2} = \sqrt{1+36} = \sqrt{37}$$

$$\text{direction} = \frac{\bar{v}(0)}{|\bar{v}(0)|} = \frac{1}{\sqrt{37}}(-\bar{i} + 6\bar{k})$$

$$\text{velocity} = \text{speed} \times \text{direction}$$

$$= \sqrt{37}\left[\frac{-\bar{i} + 6\bar{k}}{\sqrt{37}}\right]$$

Ex. If $\bar{r}(t) = \frac{4}{9}(1+t)^{3/2}\bar{i} + \frac{4}{9}(1-t)^{3/2}\bar{j} + \frac{1}{3}t\,\bar{k}$ is the position of a particle in the space at time t. Find the angle between the velocity and acc. vectors at time $t=0$.

$\Rightarrow$

$$\bar{r}(t) = \frac{4}{9}(1+t)^{3/2}\bar{i} + \frac{4}{9}(1-t)^{3/2}\bar{j} + \frac{1}{3}t\,\bar{k}$$

$$\bar{v}(t) = \frac{4}{9}\cdot\frac{3}{2}(1+t)^{\frac{1}{2}}\bar{i} + \frac{4}{9}\cdot\frac{3}{2}(1-t)^{\frac{1}{2}}(-1)\bar{j} + \frac{1}{3}\bar{k}$$

$$\bar{a}(t) = \frac{2}{3}\cdot\frac{1}{2}(1+t)^{-\frac{1}{2}}\bar{i} + \frac{2}{3}\cdot\frac{1}{2}(1-t)^{-\frac{1}{2}}\bar{j} + 0\bar{k}$$

at $t=0$

$$\bar{v}(0) = \frac{2}{3}\bar{i} - \frac{2}{3}\bar{j} + \frac{1}{3}\bar{k}$$

$$\bar{a}(0) = \frac{1}{3}\bar{i} + \frac{1}{3}\bar{j} + 0\bar{k}$$

If $\bar{u} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

$\bar{v} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

then angle betn $\bar{u}$ & $\bar{v}$ is

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$\therefore$ Angle betn velocity & acc. at $t=0$ is

$$\cos\theta = \frac{\frac{2}{3}\left(\frac{1}{3}\right) + \left(-\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)(0)}{\sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}}\sqrt{\frac{1}{9}+\frac{1}{9}+0}}$$

$$= \frac{\frac{2}{9} - \frac{2}{9} + 0}{\sqrt{2/9}} = 0$$

$$\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

$\therefore$ velocity & acc. at $t=0$ are perpendicular to each other.

Ex. Find parametric eqns for the line that is tangent to the curve

$$\bar{r}(t) = \sin(t)\bar{i} + (t^2 - \cos t)\bar{j} + e^t\bar{k} \text{ at } t_0 = 0.$$

$$\Rightarrow \quad r(0) = \sin(0)\bar{i} + (0 - \cos 0)\bar{j} + e^0\bar{k}$$

$$r(0) = 0\bar{i} - \bar{j} + \bar{k}$$

$\therefore$ point on tgt line is $(0,-1,1)$

For d.r.s. consider

$$\bar{v}(t) = \frac{d\bar{r}}{dt} = \cos t\,\bar{i} + (2t + \sin t)\bar{j} + e^t\bar{k}$$

$$\bar{v}(0) = \bar{i} + 0\bar{j} + \bar{k}$$

$\therefore$ d.r.s. of tgt are $1, 0, 1$

$\therefore$ Eqn of tgt line passing through $(0,-1,1)$ and having d.r.s. $1,0,1$ is

$$\frac{x-0}{1} = \frac{y+1}{0} = \frac{z-1}{1} = \lambda$$

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} = \lambda$$

$\therefore \; x = \lambda, \; y = -1, \; z = \lambda + 1$

Ex. Find the values of t so that the tgt line to the curve $\bar{r}(t) = 2t\bar{i} + t^2\bar{j} - t^2\bar{k}$ contains the point $(0,-4,4)$.

$\Rightarrow$ $\bar{r}(t) = 2t\bar{i} + t^2\bar{j} - t^2\bar{k}$

point on the tgt is $(2t, t^2, -t^2)$

$$\bar{v}(t) = \frac{d\bar{r}}{dt} = 2\bar{i} + 2t\bar{j} - 2t\bar{k}$$

$\therefore$ d.r.s. of tgt $2, 2t, -2t$

eqn of tgt line is

$$\frac{x-2t}{2} = \frac{y-t^2}{2t} = \frac{z+t^2}{-2t} = \lambda$$

$$x = 2\lambda + 2t, \quad y = 2t\lambda + t^2, \quad z = -2t\lambda - t^2$$

Now $(0,-4,4)$ lies on tgt

$\therefore$ $2\lambda + 2t = 0$ —— (1)

$2\lambda t + t^2 = -4$ —— (2)

$-2\lambda t - t^2 = 4$ —— (3)

from (1) $\lambda = -t$

$\therefore$ from (2)

$$2(-t)t + t^2 = -4$$

$$-2t^2 + t^2 = -4$$

$$\Rightarrow -t^2 = -4$$

$$\Rightarrow t^2 = 4$$

$$\Rightarrow t = \pm 2$$

Reference: Vector Calculus, text book for S.Y.B.Sc., golden series by Nirali Prakashan.