Matrices and System of Linear Equations

* Row echelon form -

A matrix is said to be in row echelon form if it satisfies following conditions

1) If a row does not consist entirly of zeros, then the first non zero entry should be 1. That 1 is called as leading 1.
2) If there are any rows that consist entirly zeros, then write down that row at bottom of the matrix
3) If any two successive rows that do not consist entirely zeros, the leading 1 in the lower row occurs forther to the right than the leading 1 in the higher row.
4) All the entries above the leading 1 are zero.

Ex.

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -4 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

* Reduced row echelon form -

A matrix is said to be in reduced row echelon form, if it is in row echelon form and all the entries below and above leading 1 must be zero.

Ex.

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

* System of linear equations

A system of m linear equations in n unknown can be written as

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

where $x_1, x_2, \ldots, x_n$ are unknown. In matrix form it can be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$AX = B$$

where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

*** Augmented Matrix -**

Augmented matrix of above system can be written as

$$[A|B] = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

*** Consistent -**

A system of equations is said to be consistent if it has solution.

*** Inconsistent -**

A system of equations is said to be inconsistent if it does not have solution.

Remark - Every system of linear equations has no solution, or has exactly one solution, or has infinitely many solutions.

* Let $AX=B$ be the system of linear equations

$AX=B$

$\rho(A)=\rho(A|B)$ system is consistent

- $\rho(A)=\rho(A|B)=$ no. of variables $\rightarrow$ system has unique sol^n
- $\rho(A)=\rho(A|B) <$ no. of variables $\rightarrow$ system has infinitely many solutions.

$\rho(A)\neq\rho(A|B)$ system is inconsistent

* **Gaussian Elimination Method-**

In Gaussian elimination method we reduce the augmented matrix to row echelon form and then solve system by using above conditions.

Ex. Solve the linear system by Gaussian elimination method.

1] $x_1+x_2+2x_3=9$

$2x_1+4x_2-3x_3=1$

$3x_1+6x_2-5x_3=0$

Rewriting the equation in augmented matrix form

$$[A|B]=\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

$R_2-2R_1,\ R_3-3R_1$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right]$$

$\frac{1}{2}R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{array}\right]$$

$R_3 - 3R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right]$$

$-2R_3$

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & = \end{array} \\ \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array}\right] \end{array}$$

Rewriting the system we get

$$x_1 + x_2 + 2x_3 = 9 \quad \text{---} ①$$

$$x_2 - \frac{7}{2}x_3 = \frac{-17}{2} \quad \text{---} ②$$

$$x_3 = 3$$

from ②

$$x_2 - \frac{7}{2}(3) = \frac{-17}{2}$$

$$x_2 = \frac{-17}{2} + \frac{21}{2} = \frac{4}{2} = 2$$

from ①

$$x_1 + 2 + 6 = 9$$

$$x_1 = 1$$

$\therefore x_1 = 1, \; x_2 = 2, \; x_3 = 3$

2] $x_1 + 2x_3 + x_4 = 5$

$x_1 + x_2 + 5x_3 + 2x_4 = 7$

$x_1 + 2x_2 + 8x_3 + 4x_4 = 12$

Augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 5 \\ 1 & 1 & 5 & 2 & 7 \\ 1 & 2 & 8 & 4 & 12 \end{array}\right]$$

$R_2 - R_1,\ R_3 - R_1$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 5 \\ 0 & 1 & 3 & 1 & 2 \\ 0 & 2 & 6 & 3 & 7 \end{array}\right]$$

$R_3 - 2R_2$

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & = \end{array} \\ \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 5 \\ 0 & 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \end{array}$$

$\rho(A) = \rho(A|B) = 3 <$ no. of variables

$\therefore$ It has infinitely many solutions.

Take x_3 as a free variable

$x_3 = t \quad , \quad t \in \mathbb{R}$

rewriting the system we get

$x_1 + 2x_3 + x_4 = 5 \quad$ — ①

$x_2 + 3x_3 + x_4 = 2 \quad$ — ②

$x_4 = 3$

from ②

$x_2 + 3t + 3 = 2$

$x_2 = -1 - 3t$

from ①

$x_1 + 2t + 3 = 5$

$\therefore x_1 = 2 - 2t$

$\therefore x_1 = 2 - 2t,\ x_2 = -1 - 3t,\ x_3 = t,\ x_4 = 3, \quad t \in \mathbb{R}$.

3] $x + 5y + 4z - 13w = 3$

$3x - y + 2z + 5w = 2$

$2x + 2y + 3z - 4w = 1$

Augmented matrix is

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 5 & 4 & -13 & 3 \\ 3 & -1 & 2 & 5 & 2 \\ 2 & 2 & 3 & -4 & 1 \end{array}\right]$$

$R_2 - 3R_1$, $R_3 - 2R_1$

$$\sim \left[\begin{array}{cccc|c} 1 & 5 & 4 & -13 & 3 \\ 0 & -16 & -10 & 44 & -7 \\ 0 & -8 & -5 & 22 & -5 \end{array}\right]$$

$-\frac{1}{16} R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & 5 & 4 & -13 & 3 \\ 0 & 1 & \frac{10}{16} & \frac{-44}{16} & \frac{7}{16} \\ 0 & -8 & -5 & 22 & -5 \end{array}\right]$$

$R_3 + 8R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & 5 & 4 & -13 & 3 \\ 0 & 1 & \frac{10}{16} & -\frac{44}{16} & \frac{7}{16} \\ 0 & 0 & 0 & 0 & -\frac{3}{2} \end{array}\right]$$

$$\rho(A) = 2 \;\; \& \;\; \rho(A|B) = 3$$

$$\rho(A) \neq \rho(A|B)$$

$\therefore$ Given system is inconsistent.

* Gauss-Jordan Method-

In Gauss-Jordan method we reduce the augmented matrix to reduce row echelon form. and then solve the system.

Ex. solve the following system by Gauss-Jordan method.

1) $x - 3y + 5z = -9$

$2x - y - 3z = 19$

$3x + y + 4z = -13$

Augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 2 & -1 & -3 & 19 \\ 3 & 1 & 4 & -13 \end{array}\right]$$

$R_2 - 2R_1, \; R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 5 & -13 & 37 \\ 0 & 10 & -11 & 14 \end{array}\right]$$

$\frac{1}{5}R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 1 & -\frac{13}{5} & \frac{37}{5} \\ 0 & 10 & -11 & 14 \end{array}\right]$$

$R_1 + 3R_2, \; R_3 - 10R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{14}{5} & \frac{66}{5} \\ 0 & 1 & -\frac{13}{5} & \frac{37}{5} \\ 0 & 0 & \frac{75}{5} & -\frac{300}{5} \end{array}\right]$$

$\frac{5}{75} R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{14}{5} & \frac{66}{5} \\ 0 & 1 & -\frac{13}{5} & \frac{37}{5} \\ 0 & 0 & 1 & -4 \end{array}\right]$$

$R_1 + \frac{14}{5} R_3, \; R_2 + \frac{13}{5} R_3$

$$\begin{array}{c} \begin{array}{cccc} x & y & z & = \end{array} \\ \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -4 \end{array}\right] \end{array}$$

$\rho(A) = \rho(A|B) =$ no. of variables $= 3$

$\therefore$ system has unique solution

Rewriting equation

$$x = 2$$
$$y = -3$$
$$z = -4$$

2] $\quad 2x + 2y + 4z = 0$

$\omega - y - 3z = 0$

$2\omega + 3x + y + z = 0$

$-2\omega + x + 3y - 2z = 0$

Augmented matrix is

$$[A|B] = \left[\begin{array}{cccc|c} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{array}\right]$$

$R_3 - 2R_1, \; R_4 + 2R_1$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{array}\right]$$

$\frac{1}{2} R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{array}\right]$$

$R_3 - 3R_2, \; R_4 - R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -10 & 0 \end{array}\right]$$

$R_1 + 3R_3, \; R_2 - 2R_3, \; R_4 + 10R_3$

$$\begin{array}{c} \begin{array}{ccccc} w & x & y & z & = \end{array} \\ \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$\rho(A) = \rho(A|B) = 3 <$ no. of variables $= 4$

$\therefore$ It has infinitely many solutions.

Take y as a free variable

$$y = t, \quad t \in \mathbb{R}$$

$$w - y = 0 \Rightarrow w = t$$

$$x + y = 0 \Rightarrow x = -t$$

$$z = 0$$

$$\therefore \; w = t, \; x = -t, \; y = t, \; z = 0.$$

Eg. Determine the values of a for which system has no solutions, exactly one solution, or infinitely many solutions.

$$x + 2y + z = 2$$

$$2x - 2y + 3z = 1$$

$$x + 2y - (a^2 - 3)z = a$$

Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & -2 & 3 & 1 \\ 1 & 2 & -a^2+3 & a \end{array}\right]$$

$R_2 - 2R_1, \; R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -6 & 1 & -3 \\ 0 & 0 & -a^2+2 & a-2 \end{array}\right]$$

① No solution

$$-a^2 + 2 = 0 \;\; \& \;\; a - 2 \neq 0$$

$$a^2 = 2 \;\; \& \;\; a \neq 2$$

$$\therefore \; a = \pm\sqrt{2}$$

② Unique soln

$$-a^2 + 2 \neq 0$$

$$a \neq \pm\sqrt{2}$$

③ Infinitely many solutions

$$-a^2 + 2 = 0 \;\; \& \;\; a - 2 = 0$$

$$a = \pm\sqrt{2} \;\; \& \;\; a = 2$$

which is not possible.

Reference – Linear Algebra, A textbook for S.Y.B.Sc.
Golden series by Dr. S. Gaikwad, Dr. K. Takale et. al.