Unit 3: Real Numbers

In ancient India, the concept of real numbers was intertwined with the development of mathematics and astronomy. The Sulba Sutras (circa 800 – 500 BCE) contained early references to irrational numbers in the context of geometric constructions. Indian mathematicians like Aryabhata (476 – 550 CE) and Brahmagupta (598 – 668 CE) made significant contributions, including rules for arithmetic operations with zero and negative numbers.

- **Notion of sets:**

$\mathbb{N}$ = Set of all **natural numbers** (Positive Integers) = $\{1, 2, 3, 4, 5, \ldots\}$

W = Set of all whole numbers **=** $\{0, 1, 2, 3, 4, 5, \ldots\}$

$\mathbb{Z}$ = Set of all **Integers** = $\{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$

$\mathbb{Q}$ = Set of all **Rational numbers** = $\left\{\frac{p}{q} / \, p, q \in \mathbb{Z}, q \neq 0\right\}$

$\mathbb{R}$ = Set of all **Real numbers** = $\mathbb{Q} \cup \mathbb{Q}^c$;

Where, $\mathbb{Q}^c$ is set of Irrational numbers are numbers. The real numbers which cannot be expressed as a simple fraction of two integers.

$\mathbb{C}$ = Set of **Complex numbers** =$\left\{a + ib \, / a, b \in \mathbb{R}, i = \sqrt{-1}\right\}$.

The first known use of the notion of i (the imaginary unit) was by the Italian mathematician Rafael Bombelli in his work "L'Algebra" published in 1572. He used iii to handle the square roots of negative numbers while solving cubic equations, despite the broader mathematical community initially being skeptical of imaginary numbers.

- **Real Numbers in day – to – day life:**

Studying real numbers is fundamental because they form the basis for most mathematical concepts and are essential for understanding and solving real-world problems. Real numbers allow us to measure, quantify, and analyze continuous quantities, making them crucial for fields such as science, engineering, finance, and everyday decision-making. Mastery of real numbers enables precise communication, critical thinking, and problem-solving skills that are applicable in various academic disciplines and practical situations.

- ## Algebraic properties of Real numbers or Field axioms:

Two binary operations '+' and '•', called addition and multiplication respectively satisfy the following axioms-

1. For $a, b \in \mathbb{R}, a + b \in \mathbb{R}$. **Closure ness of '+'.**

2. For $a, b \in \mathbb{R}, a + b = b + a$ **Commutativity of '+'.**

3. For $a, b, c \in \mathbb{R}, (a + b) + c = a + (b + c)$ **Associativity of '+'.**

4. For $a \in \mathbb{R}$ there is a number $0 \in \mathbb{R}$ *such that*

   $a + 0 = 0 + a = a$ **Existence of additive identity.**

5. For every $a \in \mathbb{R}$ there is a number $-a \in \mathbb{R}$ *such that* $a + (-a) = (-a) + a = 0$

   Existence of negative element /Additive inverse.

6. For $a, b \in \mathbb{R}, a \bullet b \in \mathbb{R}$. **Closureness of '•'**

7. For $a, b \in \mathbb{R}, a \bullet b = b \bullet a$ **Commutativity of '•'**

8. For $a, b, c \in \mathbb{R}, (a \bullet b) \bullet c = a \bullet (b \bullet c)$ **Associativity of '•'**

9. For $a \in \mathbb{R}$ there is a number $1 \in \mathbb{R}$ *such that*

   $a \bullet 1 = 1 \bullet a = a$ **Existence of multiplicative identity.**

10. For every $a \neq 0 \in \mathbb{R}$ there is a number $\frac{1}{a} \in \mathbb{R}$ *such that*

    $a \bullet \left(\frac{1}{a}\right) = \left(\frac{1}{a}\right) \bullet a = 1$ **Existence of multiplicative inverse.**

11. For $a, b, c \in \mathbb{R}, a \bullet (b + c) = a \bullet b + a \bullet c$ **Multiplication distributive over addition.**

Definition:

A set which satisfies all above properties is called as a **Field.**

For example

1. A set of all real numbers ($\mathbb{R}$) is a field.
2. The set of all rational numbers ($\mathbb{Q}$) is a field.
3. The set of integers is not a field

Theorem 1:

(a) If $a\ and\ z$ are any elements of $\mathbb{R}$ such that $z + a = a$ then $z = 0$.

(b) If $u\ and\ b \neq 0$ are any elements of $\mathbb{R}$ such that $u \bullet b = b$ then $u = 1$.

(c) If a is any element of $\mathbb{R}$ then $a \bullet 0 = 0 = 0 \bullet a$.

(d) If $a \neq 0\ and\ b$ in $\mathbb{R}$ such that $a \bullet b = 1\ then\ b = \frac{1}{a}$.

(e) If $a \bullet b = 0$ then either $a = 0\ or\ b = 0\ or$ both are zero.

(f) If a, b are in $\mathbb{R}$ such that $a + b = 0\ then\ b = -a$.

Proof:

(a) It is given that $z + a = a$

we add $(-a)$ on both sides of the above equation; which gives us

$(z + a) + (-a) = a + (-a)$

$\Longrightarrow z + \big(a + (-a)\big) = 0$, by axiom (Associativity of '$+$')

$\therefore z + 0 = 0,$ by axiom (Existence of additive inverse)

$\therefore z = 0.$ by axiom (Existence of additive identity)

(b) Since $b \neq 0$ there exist $\frac{1}{b}$ in $\mathbb{R}$ such that $b \bullet \frac{1}{b} = 1$.

We are given that $u \bullet b = b$.

Multiply both sides by 1/b.

$\therefore (u \bullet b) \bullet \frac{1}{b} = b \bullet \frac{1}{b}$

$\therefore u \bullet \left(b \bullet \frac{1}{b}\right) = 1$, by axiom (8) and (10)

$\therefore u \bullet 1 = 1$, by axiom (10)

$\therefore u = 1$. By axiom (9)

(c) We know that, $1 + 0 = 1$ by axiom (4)

Multiply by 'a' on both sides of the above equation and using axiom (11)

It gives us; $a \bullet 1 + a \bullet 0 = a \bullet 1$

$\Longrightarrow a \bullet 1 + a \bullet 0 = a \bullet 1$

$\Longrightarrow a + \ a \bullet 0 = a$ by axiom (9)

Now, add $(-a)$ on both sides of above equation we will get

$(-a) + a + a \bullet 0 = (-a) + a$

$\Rightarrow \big(a + (-a)\big) + a \bullet 0 = 0$

$\Rightarrow 0 + a \bullet 0 = 0$

$\Rightarrow a \bullet 0 = 0.$

Similarly; $0 \bullet a = 0$

(d) Since $a \neq 0$ by axiom (10) there exist $\frac{1}{a} \in \mathbb{R}$ such that $a \bullet \frac{1}{a} = 1.$

We know that, $b = 1 \bullet b$ by axiom (9)

$= \left(a \bullet \frac{1}{a}\right) \bullet b$, by above

$= \left(\frac{1}{a} \bullet a\right) \bullet b$, by axiom (7)

$= \frac{1}{a} \bullet (a \bullet b)$, by axiom (8)

$= \frac{1}{a} \bullet 1$, by given hypothesis that $a \bullet b = 1$

$\therefore b = \frac{1}{a}$. By axiom (9)

(e) Suppose $a \neq 0$ then there exists $\frac{1}{a} \in \mathbb{R}$ *such that* $a \bullet \frac{1}{a} = 1.$

Now, we have given that $a \bullet b = 0$ Multiply both sides by $\frac{1}{a}$.

$\frac{1}{a} \bullet (a \bullet b) = \frac{1}{a} \bullet 0$

$\Rightarrow \left(\frac{1}{a} \bullet a\right) \bullet b = 0$, by axiom (8) and (9)

$\Rightarrow 1 \bullet b = 0$ by axiom (10)

$\Rightarrow b = 0$. by axiom (9)

Similarly, we can prove that if $b \neq 0$ *then* $a = 0.$

(f) We have given that $a + b = 0$.

Adding $(-a)$ in both sides, we get

$(-a) + (a + b) = (-a) + 0$

$\therefore \big((-a) + a\big) + b = -a$ by axiom (3) and (4)

$\therefore 0 + b = -a$ by axiom (5)

$\therefore b = -a$ by axiom (4)

Theorem 2: Let $a, b \in \mathbb{R}$ then

(a) The equation $a + x = b$ has unique solution $x = (-a) + b$ in $\mathbb{R}$.

(b) If $a \neq 0$ then the equation $a \bullet x = b$ has unique solution $x = \left(\frac{1}{a}\right) \bullet b$ in $\mathbb{R}$.

Proof:

(a) We have given that $a + x = b$.

Adding $(-a)$ in both sides,

$(-a) + (a + x) = (-a) + b$

$\therefore \big((-a) + a\big) + x = (-a) + b$ by axiom (3)

$\therefore 0 + x = (-a) + b$ by axiom (4)

$\therefore x = (-a) + b$, which is a solution.

To show the **uniqueness**, suppose there are two solutions (say) x_1, x_2 .

$\therefore a + x_1 = b \text{ } and \text{ } a + x_2 = b$. So, we need to show that $x_1 = x_2$.

Subtracting the above equations, we get

$x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2$.

(b)

We have given that $a \bullet x = b$ ...... (i)

$and \text{ } a \neq 0 \ \therefore \ \exists \ \frac{1}{a} \in \mathbb{R} \text{ } such \text{ } that \text{ } a \bullet \frac{1}{a} = 1$.

Multiply both sides of equation (i) by $\frac{1}{a}$.

$\therefore \frac{1}{a} \bullet (a \bullet x) = \left(\frac{1}{a}\right) \bullet b$

$\therefore \left(\frac{1}{a} \bullet a\right) \bullet x = \left(\frac{1}{a}\right) \bullet b$

$\therefore 1 \bullet x = \left(\frac{1}{a}\right) \bullet b$

$\therefore x = \left(\frac{1}{a}\right) \bullet b$, which is the required solution.

Now, for **uniqueness**, suppose there are two solutions (say) x_1, x_2

$\therefore a \bullet x_1 = b \; and \; a \bullet x_2 = b.$

Here, we need to show that $x_1 = x_2$.

Subtracting these equations, we will get

$a \bullet (x_1 - x_2) = 0$

$\Rightarrow x_1 - x_2 = 0, \; since \; a \neq 0 \; given$

$\Rightarrow x_1 = x_2.$

Theorem 3: If $a \in \mathbb{R}$ then

(a) $(-1) \bullet a = -a$

$(b) - (-a) = a$

$(c) \; (-1) \bullet (-1) = 1$

Proof: Left as an Exercise

Theorem 4: Let $a, b, c \in \mathbb{R}$ then

$(a) \; If \; a \neq 0 \; then \; \frac{1}{a} \neq 0 \; and \; \frac{1}{\frac{1}{a}} = a$

$(b) \; If \; a \bullet b = a \bullet c \; and \; a \neq 0 \; then \; b = c.$

Proof: Left as an Exercise

Example 1:

For any $a, b \in \mathbb{R}$ prove that

(i) If $a + b = 0$ $then$ $b = -a$. (ii)$(-1) \cdot \mathrm{a} = -\mathrm{a}$.

Solution:

(i) Suppose,$a + b = 0$, adding $(-a)$ to both sides, we get $(-a) + (a + b) = (-a) + 0$

$\therefore L.H.S. = \big((-a) + a\big) + b, by\ axiom\ (3)$

$= 0 + b, by\ axiom\ (5)$

$= b, by\ axiom\ (4)$ ……. (*)

$Now, R.H.S. = (-a) + 0 = -a$ ……. (*)

From (*) and (*), $b = -a$.

(ii) We know that $0 = 0 \cdot a$.

Also, $(-1) + 1 = 0$…. (a) and $1 \cdot a = a$ ….. (b)

∴ Consider,

$0 = 0 \cdot a = \big(1 + (-1)\big) \cdot a, by\ (a)$

$= 1 \cdot a + (-1) \cdot a, by\ axiom\ (11)$

$= a + (-1) \cdot a, by\ (b)$

Adding $(-a)$ to both sides, we get

$-a + 0 = -a + a + (-1)a$

$= (-a + a) + (-1)a$

$\therefore (-a + a) + (-1)a = -a + 0$

$\therefore 0 + (-1)a = -a + 0, by\ axiom\ (5)$

$\Rightarrow (-1)a = -a$, by axiom (4).

Example 2:

If $a \in \mathbb{R}\ s.t. a \cdot a = a$ then prove that either $a = 0\ or\ a = 1$.

Solution:

Suppose $a \in \mathbb{R}\ \ s.t.\ \ a \cdot a = a$

$\therefore a^2 = a$

$\therefore a^2 - a = 0$

$\therefore a(a-1) = 0$

$\therefore$ by Theorem 1(e), $a = 0\ or\ a - 1 = 0$

$\therefore a = 0\ or\ a = 1$.

- **Order Properties of $\mathbb{R}$**

Property 1: Closurness

Suppose S is a non-empty subset of $\mathbb{R}^+$

O_1: $If\ a, b \in S$ then $a + b \in S$.

O_2: $If\ a, b \in S\ then\ a \cdot b \in S$.

Property 2: Trichotomy Property

If $a \in \mathbb{R}$ then exactly one of the following holds $a \in \mathbb{R}^+, a = 0\ , -a \in \mathbb{R}^-$.

This property divides the set of real numbers $\mathbb{R}$ into three sets (which are subsets of $\mathbb{R}$) viz. Set of all negative real numbers, set of all positive real numbers and set containing only 0 element. Thus, Set of real numbers is the union of three disjoint sets. i. e. $\mathbb{R} = \{\mathbb{R}^-\} \cup \{\mathbf{0}\} \cup \{\mathbb{R}^+\}$.

(i) If $a \in \mathbb{R}\ and\ a > 0$ then we say that a is positive (strictly positive) real number.

(ii) If $a \in \mathbb{R}^+ \cup \{0\}\ we\ write\ it\ as\ a \geq 0$, say that a is non-negative real number.

(iii) If $-a \in \mathbb{R}$ we write $a < 0$, say that a is a negative (strictly negative) real number.

(iv) If $-a \in \mathbb{R}^- \cup \{0\}$, we write $a \leq 0$, say that a is a non-positive real number.

Property: Law of Trichotomy

If a and b are elements of $\mathbb{R}$ then exactly one of the following is true either $a < b$ or $a = b$ or $a > b$.

Result: Suppose that a and b are real numbers then

(i) If $a - b \in \mathbb{R}^+$ then $a > b$ or $b < a$.
(ii) If $a - b \in \mathbb{R}^+ \cup \{0\}$ then $a \geq b$ or $b \leq a$.

Definition: The statement which involves order relation is called an **Inequality.**

Theorem 5:

Suppose a, b, c are any elements of $\mathbb{R}$.

(i) If $a > b$ and $b > c$ $then$ $a > c$.

(ii) If $a > b$ $then$ $a + c > b + c$.

(iii) If $a > b$ and $c > 0$ $then$ $ca > cb$.

(iv) If $a > b$ and $c < 0$ $then$ $ca < cb$.

Proof:

(i) As $a > b$ and $b > c$

$\therefore a - b, b - c \in \mathbb{R}$ $then$ by $O_{1,}$

$(a - b) + (b - c) = a - c \in \mathbb{R}^+$

$\Rightarrow a > c$.

(ii) As $a > b$

$\therefore a - b \in \mathbb{R}$ but $a - b = (a + c) - (b + c) \in \mathbb{R}^+$

$\therefore a + c > b + c$.

(iii) As $a > b$ and $c > 0$

$\therefore by\ O_2,\ c(a - b) = ca - cb \in \mathbb{R}^+$

$\Rightarrow ca > cb$.

(iv) As $a > b$ and $c < 0$

$\therefore by\ O_2, (-c)(a - b) = cb - ca \in \mathbb{R}^+$

$\Rightarrow cb > ca$ $i.e.$ $ca < cb$.

Theorem 6:

(i) If $a \in \mathbb{R}$ and $a \neq 0$ $then$ $a^2 > 0$.

(ii) $1 > 0$.

(iii) If $n \in \mathbb{N}$ $then$ $n > 0$.

Proof:

(i) As $a \neq 0$ $\therefore$ by Trichotomy Property, $a > 0$ or $a < 0$.

Now, if $a > 0$ then by O_2, $a^2 = a \bullet a > 0$.

Again if $a < 0$ $then$ $-a > 0$ $\therefore a^2 = (-a) \bullet (-a) > 0$.

(ii) Since $1 = 1^2 > 0$ by (i) $above$, $1 > 0$.

(iii) We will use Mathematical Induction on n.

Step 1: Take $n = 1$ $then$ by (ii) $above$ $1 > 0$.

Step 2: Assume that result is true for $n = k$. i. e. $k > 0$

Step 3: To prove the result for $n = k + 1$

Now, from Step 1 and Step 2: $1 > 0$ and $k > 0$

$\therefore$ by O_1, $k + 1 \in \mathbb{R}^+$ and $k + 1 > 0$.

Hence, by Mathematical Induction the result is true for all $n \in \mathbb{N}$.

Theorem 7:

If $ab > 0$ $then$ $either$

(i) $a > 0$ and $b > 0$ or

(ii) $a < 0$ and $b < 0$.

Proof:

Here, note that $ab > 0 \Rightarrow a \neq 0$ and $b \neq 0$. By law of Trichotomy, either $a > 0$ or $a < 0$.

(i) If $a > 0$ $then$ $\frac{1}{a} > 0$ $\therefore b = \left(\frac{1}{a}\right)(ab) > 0$.

(ii) If $a < 0$ $then$ $\frac{1}{a} < 0$ $\therefore b = \left(\frac{1}{a}\right)(ab) < 0$.

Corollary: If $ab < 0$ *then either*

$(i)\ a < 0$ *and* $b > 0$ *or*

$(ii)\ a > 0$ *and* $b < 0$.

Example 3: Determine the set A of all real numbers x *such that* $2x - 3 \leq 6$.

Solution: For $x \in A$, *we have* $2x - 3 \leq 6$,

Adding '3' in both sides we have, $2x \leq 9$.

$\therefore x \leq \frac{9}{2}$. Hence, the required set A is $A = \left\{x \in \mathbb{R}: x \leq \frac{9}{2}\right\}$.

Example 4: Determine the set $A = \{x \in \mathbb{R}: x^2 + x > 2\}$.

Solution: For $x \in A$, *we have* $x^2 + x > 2\ \therefore x^2 + x - 2 > 0$

$\therefore (x-1)(x+2) > 0$..... (*)

Case (i) $(x - 1) > 0$ *and* $(x + 2) > 0 \therefore x > 1$ *and* $x > -2$.

But the (*) is true only for $x > 1$.

Case (ii) $(x - 1) < 0$ *and* $(x + 2) < 0 \therefore x < 1$ *and* $x < -2$.

But (*) is true only when $x < -2$.

Hence, the required set A is $A = \{x \in \mathbb{R}: x > 1\} \cup \{x \in \mathbb{R}: x < -2\}$.

Example 5: Determine the set $A = \{x \in \mathbb{R}: x^2 > 3x + 4\}$.

Solution: For $x \in A$, *we have* $x^2 > 3x + 4\ \therefore x^2 - 3x - 4 > 0$

$\therefore (x-4)(x+1) > 0$..... (*)

Case (i) $(x - 4) > 0$ *and* $(x + 1) > 0$

$\therefore x > 4$ *and* $x > -1$. But the (*) is true only for $x > 4$.

Case (ii) $(x - 4) < 0$ *and* $(x + 1) < 0$

$\therefore x < 4$ *and* $x < -1$.

But (*) is true only when $x < -1$.

Hence, the required set A is $A = \{x \in \mathbb{R}: x > 4\} \cup \{x \in \mathbb{R}: x < -1\}$.

Property: Bernoulli's inequality-

Statement: If $x > -1$ then $(1+x)^n \geq 1 + nx, for\ all\ n \in \mathbb{N}$.

Proof: We shall prove this result by Mathematical Induction on n.

Step 1: Take $n = 1\ then\ L.H.S. = 1 + x \geq 1 + x = R.H.S.$

Step 2: Assume that result is true for $n = k, k > 1$. i. e.$(1+x)^k \geq 1 + kx$.

Step 3: To prove the result for $n = k + 1$

Now, $(1+x)^{k+1} = (1+x)^k(1+x)$

$\geq (1+kx)(1+x)$

$= kx^2 + (k+1)x + 1$

$\geq 1 + (k+1)x$.

$\Rightarrow (1+x)^{k+1} \geq 1 + (k+1)x$

i. e. the result is true for $n = k + 1$.

Hence by mathematical induction the result is true for all $n \in \mathbb{N}$.

Theorem 8: If a, b are any elements $of\ \mathbb{R}\ and\ if\ a < b\ then\ a < \frac{1}{2}(a+b) < b$.

Proof:

Since $a < b$ adding a in both sides,

$a + a = 2a < a + b$. ….. (i)

Also adding b in both sides,

$a + b < b + b = 2b$. ….. (ii) From (i) and (ii) we will get,

$\therefore 2a < a + b < 2b$.

Hence, $a < \frac{1}{2}(a+b) < b$.

Corollary: If $b \in \mathbb{R}\ and\ b \neq 0\ then\ 0 < \frac{1}{2}b < b$.

Proof: Take $a = 0$ in above theorem.

Definition: If $a, b > 0 \in \mathbb{R}$ then their Arithmetic Mean and Geometric Mean is given by

$$A.M. = \frac{1}{2}(a+b) \text{ and } G.M. = \sqrt{a.b}$$

Theorem 9: If $a, b > 0 \in \mathbb{R}$ then $\sqrt{a.b} \leq \frac{1}{2}(a+b)$ and equality holds $\Leftrightarrow a = b$.

Proof:

Step I] Since $a > 0, b > 0$ *and* $a \neq b$.

$\Rightarrow \sqrt{a} > 0, \sqrt{b} > 0$ *and* $\sqrt{a} \neq \sqrt{b}$.

$\therefore \left(\sqrt{a} - \sqrt{b}\,\right)^2 > 0$

$\therefore a - 2\sqrt{a}.\sqrt{b} + b > 0$

$\Rightarrow a - 2\sqrt{a.b} + b > 0$

$\Rightarrow \sqrt{a.b} \leq \frac{1}{2}(a+b)$.

Now, if $a = b(> 0)$ then both sides of inequality equal to a. Therefore it becomes an equality, which proves that inequality holds for $a > 0, b > 0$.

Step II] Conversely, Suppose that $a > 0, b > 0$ *and* $\sqrt{a.b} = \frac{1}{2}(a+b)$. Here, squaring both sides and multiplying by 4.

$\therefore 4a.b = (a+b)^2 = a^2 + 2a.b + b^2$

$\therefore a^2 - 2a.b + b^2 = 0.$

$\Rightarrow (a-b)^2 = 0 \Leftrightarrow a = b.$

Absolute Value of a real number:

Definition: The absolute value of a real number x is denoted by $|x|$ and is defined as-

$$|x| = \begin{cases} x, & if\ x > 0; \\ 0, & if\ x = 0; \\ -x, & if\ x < 0. \end{cases}$$

Note that: Absolute value of a real number is never negative i. e. it is always positive. This is because, absolute value of a real numbers gives us the distance of that number from '0' on real line.

Theorem 10:

(i) $|a.b| = |a||b|,\ for\ all\ a, b\ in\ \mathbb{R}$.

(ii) $|a|^2 = |a^2| = a^2, for\ all\ a\ in\ \mathbb{R}$.

(iii) If $c \geq o\ then\ |x| \leq c\ if\ and\ only\ if - c \leq x \leq c$.

(iv) $-|x| \leq |x| \leq |x|, for\ all\ x\ in\ \mathbb{R}$.

Proof:

(i) If either $a\ or\ b$ is equal to 0 then the result is true.

Case (a) If $a > 0\ and\ b > 0\ then\ ab > 0$

$\therefore\ |a.b| = ab = |a||b|$.

Case (b) If $a < 0\ and\ b < 0\ then\ ab > 0$

$\therefore\ |a.b| = ab = (-a).(-b) = |a||b|$.

Case (c) If $a > 0\ and\ b < 0\ then\ ab < 0$

$\therefore\ |a.b| = -ab = a.(-b) = |a||b|$.

Case (d) If $a < 0\ and\ b > 0\ then\ ab < 0$

$\therefore\ |a.b| = -ab = (-a).b = |a||b|$.

Thus in all cases the equality holds good.

(ii) Since $a^2 > 0, we\ have\ |a^2| = |a.a| = |a||a| = |a|^2 = a^2$. i. e. $|a|$ is the non-negative square root of a^2.

(iii) If $|x| \leq c$ then we have by definition of absolute value, $x \leq c\ and - x \leq c\ i.e.$

$x \geq -c\ and\ x \leq c. \therefore\ -c \leq x \leq$ c.

Conversely, if $-c \leq x \leq c\ then\ we\ have, x \leq c\ and - c \leq x$

$i.e. x \leq c\ and - x \leq c$

$\Rightarrow |x| \leq c$. Hence the result is proved.

(iv) Put $c = |x|$ in above (iii) we will get, $-|x| \leq |x| \leq |x|$.

Theorem 11: Triangle inequality:

Statement: $If\ a, b\ in\ \mathbb{R}\ then\ |a + b| \leq |a| + |b|$.

Proof: We have $-|a| \leq |a| \leq |a|\ \ and\ \ -|b| \leq |b| \leq |b|$.

Adding these we will get,

$-(|a| + |b|) \leq (a + b) \leq (|a| + |b|)$.

Using Theorem 10 (iv), we get $|a + b| \leq |a| + |b|$.

Corollary: $If\ a, b\ in\ \mathbb{R}\ \ then$

$(i)\big||a| - |b|\big| \leq |a - b|$ $\qquad (ii)|a - b| \leq |a| + |b|$.

Proof:

(i) We write $a = (a - b) + b.\ Taking\ absolute\ value,$

$|a| = |(a - b) + b| \leq |a - b| + |b|$

$\Rightarrow\ |a| - |b| \leq |a - b|$...... (*)

Now, $b = (b - a) + a$. Taking absolute value,

$|b| = |(b - a) + a| \leq |b - a| + |a|$

$= -|b - a| + |a| = -|a - b| + |a|$.

$\Rightarrow -|a - b| \leq |a| - |b|$....... (**)

From inequality (*) and (**), we will get,

$\big||a| - |b|\big| \leq |a - b|$.

(ii) Replace b by (-b) in triangle inequality,

$\therefore |a + (-b)| \leq |a| + |-b|$

$\therefore |a - b| \leq |a| + |b|,\ Since\ |-b| = |b|$.

Corollary: If $a_1, a_2, a_3, \dots, a_n \in \mathbb{R}\ then$

$|a_1 + a_2 + a_3 + \cdots + a_n| \leq |a_1| + |a_2| + |a_3| + \cdots + |a_n|$.

Theorem 12: $If\ a, b \in \mathbb{R}$ then $|a+b| = |a| + |b|$ if and only if $a.b \geq 0$.

Proof:

Part I:

Let us suppose that $|a+b| = |a| + |b|$ for all $a, b \in \mathbb{R}$.

Consider, $|a+b|^2 = (a+b)^2 = a^2 + 2a.b + b^2$..... (i)

and $(|a| + |b|)^2 = a^2 + b^2 + 2|a||b|$..... (ii)

From (i) and (ii), $|a||b| = ab\ i.e. |a.b| = a.b$..... (iii)

Now, on the contrary assume that $ab < 0\ \therefore |ab| = -a.b$. With this, equation (iii) becomes $2a.b = 0 \Rightarrow a.b = 0$

$\Rightarrow either\ a = 0\ or\ b = 0$. This gives the contradiction to the fact that $a.b < 0$.

Hence, $a.b \geq 0$.

Part II (Converse):

Suppose that $a.b \geq 0$.

Case 1: $a.b = 0 \Rightarrow a = 0\ or\ b = 0$.

$Let\ a = 0\ then\ |a+b| = |0+b| = |0| + |b| = |b|$.

Case 2: $a.b > 0 \Rightarrow either\ a > 0, b > 0\ or\ a < 0, b < 0$.

Here, in both the cases we will get, $|a+b| = |a| + |b|$.

Theorem 13: If $a, b \in \mathbb{R}\ and\ b \neq 0\ then\ |a| = \sqrt{a^2}$.

Proof:

If $a \geq 0\ then\ |a| = a\ and\ \sqrt{a^2} = a\ \therefore\ |a| = \sqrt{a^2}$.

If $a < 0\ then\ |a| = -a\ and\ \sqrt{a^2} = \sqrt{(-a)(-a)} = -a$

$\therefore\ here\ also\ |a| = \sqrt{a^2}$.

Thus, in both the cases we get, $|a| = \sqrt{a^2}$.

Note that: $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}\ , if\ y \neq 0$.

Theorem 14: $For\ x, y, a, b \in \mathbb{R}\ and\ |a| \neq |b|, \left|\frac{x+y}{a+b}\right| \leq \frac{|x|+|y|}{||a|-|b||}$.

Proof:

We know that $0 \leq |x+y| \leq |x| + |y|\ also$

$0 < ||a| - |b|| \leq |a+b|$.

Again if $0 \leq a < b\ and\ 0 < c \leq d\ then\ ac < bd$.

$\therefore |x+y|||a|-|b|| \leq (|x|+|y|)|a+b|$

$\Rightarrow \frac{|x+y|}{|a+b|} \leq \frac{|x|+|y|}{||a|-|b||}$.

Examples: Find real number x in set A which satisfy

(i)$|2x+5| < 9$ (ii) $|2x-1| \leq 13$ (iii) $A = \{x \in \mathbb{R}: |x-3| < |x|\}$

(iv) $|x-1| > |x+1|$ (v) $\left|\frac{2+x}{3+x}\right| < 1$ (vi) $|3x+4| < |x+2|$

(vii) $|x-2| + |x| = 4$ (viii) $|x+1| + |x-2| = 7$ (ix) $|x| + |x+1| < 2$

(x) $|x^2 - 1| \leq 4$.

Solution:

(i) For $x \in A \Leftrightarrow -9 < 2x + 5 < 9$

$\Leftrightarrow -14 < 2x < 4$

$\Leftrightarrow -7 < x < 2$.

$\therefore A = \{x \in \mathbb{R}: -7 < x < 2\}$.

(ii) For $x \in A \Leftrightarrow -13 < 2x - 1 < 13$

$\Leftrightarrow -12 < 2x < 14$

$\Leftrightarrow -6 < x < 7$.

(iii) We know that if $a \geq 0, b \geq 0\ then\ a < b \Leftrightarrow a^2 < b^2$.

Also $|a|^2 = a^2, as\ a^2 \geq 0, \forall\ a \in \mathbb{R}$.

Now, $|x-3| < |x| \Leftrightarrow |x-3|^2 < |x|^2$

$\therefore x^2 - 6x + 9 < x^2$

$\Leftrightarrow -6x + 9 < 0$

$\Leftrightarrow -6x < -9$

$\Leftrightarrow x > \frac{9}{6}.$

$\Leftrightarrow x > \frac{3}{2}.$

(v) $\left|\frac{2+x}{3+x}\right| < 1 \Leftrightarrow |2+x| > |3+x|$

$\Leftrightarrow (2+x)^2 > (3+x)^2$

$\Leftrightarrow 4 + 4x + x^2 > 9 + 6x + x^2$

$\Leftrightarrow -5 > 2x$

$\Leftrightarrow \frac{-5}{2} > x \; i.e. \; x < \frac{-5}{2}.$

(vi) $|3x+4| < |x+2| \Leftrightarrow (3x+4)^2 < (x+2)^2$

$\Leftrightarrow 8x^2 + 20x + 12 < 0$

$\Leftrightarrow 2x^2 + 5x + 3 < 0$

$\Leftrightarrow (2x+3)(x+1) < 0$

$\Leftrightarrow -\frac{3}{2} < x < -1.$

(vii) Squaring both sides, we will get

$(|x-2| + |x|)^2 = 16$

$\Leftrightarrow |x-2|^2 + |x|^2 + 2|x-2||x| = 16.$

$\Leftrightarrow (x-2)^2 + x^2 + 2x(x-2) = 16$

$\Leftrightarrow 4x^2 - 8x = 12$

$\Leftrightarrow x^2 - 2x - 3 = 0$

$\Leftrightarrow (x+1)(x-3) = 0 \Leftrightarrow x = -1, x = 3.$

Hence, the set $A = \{-1, 3\}$.

(ix) Squaring, we will get

$x^2 + 2x(x+1) + x^2 + 2x + 1 < 4 \Leftrightarrow 4x^2 + 4x - 3 < 0$

$\Leftrightarrow (2x-1)(2x+3) < 0$

$\Leftrightarrow 2x - 1 < 0 \; and \; 2x + 3 > 0$

$\Leftrightarrow x < \frac{1}{2}$ *and* $x > -\frac{3}{2}$

i.e. $-\frac{3}{2} < x < \frac{1}{2}$. Hence, the set $A = \left\{x \in \mathbb{R}: -\frac{3}{2} < x < \frac{1}{2}\right\}$.

(x) $-4 \leq x^2 - 1 \leq 4 \Leftrightarrow -3 \leq x^2 \leq 5$.

Case 1: $-3 \leq x^2 \Leftrightarrow x^2 + 3 \geq 0$

$\Leftrightarrow x^2 \geq -3$

$\Leftrightarrow x \geq 0$

Case 2: $x^2 \leq 5 \Leftrightarrow (x - \sqrt{5})(x + \sqrt{5}) \leq 0$

$\Leftrightarrow x - \sqrt{5} \leq 0$ *and* $x + \sqrt{5} \geq 0$

$\Leftrightarrow x \leq \sqrt{5}$ *and* $x \geq -\sqrt{5}$.

i. e. $-\sqrt{5} \leq x \leq \sqrt{5}$.

Hence, $A = \left\{x \in \mathbb{R}: -\sqrt{5} \leq x \leq \sqrt{5}\right\}$.

- **Geometrical Significance of $|x| \leq C$:**

The absolute value of a real number x ($|x|$), **Geometrically** means "the distance of x from the origin". Hence, $|x| \leq C$, **Geometrically** means "the real number x whose distance from the origin is less than or equal to C".

The **distance** between two elements/numbers a *and* b *or* x *and* y *in* $\mathbb{R}$ *is*

$|a - b|$ *or* $|x - y|$.

Definitions:

Let S be any non-empty subset of $\mathbb{R}$. Then

(i) A real number "b" is said to be **maximum element** of S if b$\in S$ and $x \leq b, \forall x \in S$. This is denoted by $\boldsymbol{b = Max\ S}$.

(ii) A real number "a" is said to be **minimum element** of S if a $\in S$ and $a \leq x, \forall x \in S$. This is denoted by $\boldsymbol{a = Min\ S}$.

Theorem 15: A maximum (minimum) element if it exists is unique.

Proof: Let if possible, b $and\ b_1$ are two maximum elements for a set S.

By the definition, $b_1 \leq b$ .... (1) as b is Max.

$b \leq b_1$... (2) as $b_1$is Max.

Therefore from (1) and (2) we will get $b = b_1$.

Similarly, we can prove for minimum element.

Definitions:

Let S be any non-empty subset of $\mathbb{R}$. Then

(a) The set S is said to be **bounded above** if there exists a number $v \in \mathbb{R}$ such that $x \leq v, for\ all\ x \in S$. Here, v is called an **upper bound** of S.

(b) The set S is said to be **bounded bellow** if there exists a number $u \in \mathbb{R}$ such that $u \leq x, for\ all\ x \in S$. Here, u is called a **lower bound** of S.

(c) The set S is said to be **bounded** if it is both bounded above and bounded bellow.

A set is said to be unbounded if it is not bounded.

Examples:

1. $T = [2, 5.5)\ i.e.\ 2 \leq x < 5.5, \forall x \in T$.

Here, every number is less than 5.5 . Hence it is an upper bound of T and every number graeter than or equal to 2. Hence it is lower bound of T.

⇒T is bounded above as well as bounded below. Hence, T is bounded set.

2. $S = (-5, 11]\ i.e. -5 < x \leq 11, \forall x \in S$.

Here, every number is less than or equal to 11. Hence it is an upper bound of S and every number graeter than -5. Hence it is lower bound of S.

⇒ S is bounded above as well as bounded below.

Hence, S is bounded set.

3. The set $\mathbb{R}^-$ is bounded above but unbounded below.

4. The set $\mathbb{R}^+$ is unbounded above but bounded below.

5. The set $\mathbb{R}$ is unbounded set.

- **W. O. P. (Well Ordering Principle):**

Every non-empty subset of set of natural numbers has a minimum (least) element. i. e. if $S\ (\neq \emptyset) \subseteq \mathbb{N}\ then\ \exists\ m \in S\ such\ that\ m \leq K, for\ all\ K \in S$.

Supremum and Infimum of a set:

Definition:

(a) Let S be any non-empty subset of $\mathbb{R}$. A real number $\boldsymbol{M}$ is called the **least upper bound or Supremum (l. u. b.)** for set S if ;

(i) M is an upper bound for S.

(ii) no number less than M is an upper bound for S. i. e for each $\varepsilon > 0$ the number $M - \varepsilon$ is not an upper bound for S. It is denoted by $\boldsymbol{M = SupS}$.

(b) Let S be any non-empty subset of $\mathbb{R}$. A real number $\boldsymbol{m}$ is called the **greatest lower bound or Infimum (g. l. b)** for set S if;

(i) m is a lower bound for S.

(ii) no number greater than m is an upper bound for S. i. e for each $\varepsilon > 0$ the number $m + \varepsilon$ is not an upper bound for S. It is denoted by $\boldsymbol{m = InfS}$.

Theorem 16: The Supremum (Infimum) for S is unique if it exists.

Proof:

Let if possible assume that S have two Suprema M and M'.

If $M = M'$. then we are through.

Therefore assume that $M \neq M'$.

$\Rightarrow$ either $M < M'\ or\ M > M'$.

Now, suppose $M < M'$. Since M' is a Supremum of S, By definition, M is not an upper bound for S which is a contradiction to our assumption that M is Supremum of S.

Again if $M' < M$, since M is Supremum, by definition, M' is not an upper bound for S, which is a contradiction to our assumption that M' is a Supremum of S.

$$\therefore M = M'.$$

Similarly, we can prove that the Infimum for the set S is unique if it exists.

Theorem 17. If $InfS$ and $SupS$ for a set S exists then

$$\boldsymbol{InfS \leq x \leq SupS, \forall x \in S. or\ g.l.b. \leq S \leq l.u.b.}$$

Examples:

1. Let $A = \{2,4,6,8\}$. $\text{Sup(A)} = 8, \text{Inf(A)} = 2$.

2. $I = [4,9)$. $Sup(I) = 9, Inf(I) = 4$.

Here, $Sup(I) \notin I\ but\ Inf\ (I) \in I$.

3. $B = (-2,5]. Sup(B) = 5, Inf(B) = -2$. Here, $Sup(B) \in B$ but $Inf(B) \notin B$.

4. $Z = (-3,7). Sup(Z) = 7, Inf(Z) = -3$. Here, $\text{Sup}(Z) \notin Z\ and\ also\ Inf(Z) \notin Z$.

Note:

For a non-empty set $S\ of\ \mathbb{R}$ about the Sup. and Inf. there are four possibilities viz.

(i) Set S can have both Sup. and Inf.

(ii) Set S can have a Sup. but not Inf.

(iii) Set S can have Inf. but not Sup.

(iv) Set S have neither Sup. nor Inf.

In general,

[1] The set $S = \{x: a \leq x \leq b\}$ i. e. Closed interval has Inf. as well as Sup. Moreover, both are in S.

[2] The set $S = \{ x: a < x < b\}$ i. e. Open interval has Inf. and Sup. but both they are not in S.

- ## The Completeness Axiom/Property of $\mathbb{R}$:

Every non – empty set of real numbers that is bounded above has a least upper bound (supremum).

Application:

[1] In Economics, the concept of a supremum is used in utility theory and the analysis of consumer preferences. The completeness property ensures the existence of an optimal consumption bundle.

[2] In Engineering, the design of systems often involves optimizing certain parameters subject to constraints. The completeness property ensures that optimal solutions exist when working within bounded regions.

Remark:

(1) The set of all real number $\mathbb{R}$ is a complete ordered field.

(2) The set of all rational numbers $\mathbb{Q}$ is not complete.

For, $E = \{x \in \mathbb{Q} : 0 \leq x, x^2 < 2\}$. Then E is a non-empty subset of $\mathbb{Q}$. The Supremum of E is $\sqrt{2}$ *but* $\sqrt{2} \notin \mathbb{Q}$, *as it is not a rational number*. Therefore, for every non empty subset of $\mathbb{Q}$ has no Sup. in $\mathbb{Q}$. i. e. $\mathbb{Q}$ does not satisfy completeness property.

Hence, $\mathbb{Q}$ is a field but not complete ordered field.

- **Archimedean Property**:

Statement: If $x, y \in \mathbb{R}, x > 0$ *then for any* $y \in \mathbb{R}\ \exists\ n \in \mathbb{N}$ *such that* $nx > y$.

Proof:

If $y \leq 0$ then the theorem is obvious. Now, if $y > 0$. To show that; $\exists\ n \in \mathbb{N}$ *such that* $nx > y$. Then on the contrary assume that $nx < y, \forall n \in \mathbb{N}$.

Consider, the set $S = \{nx : n \in \mathbb{N}\} \subseteq \mathbb{N}$ is a non-empty set, which is bounded above. Hence by Completeness axiom, S has a Supremum. Let it be M. i. e. M = Sup(S).

Therefore, by the definition of Sup(S), $nx \leq M, \forall n \in \mathbb{N}$

$$\Rightarrow (n+1)x \leq M,, \forall n \in \mathbb{N}.$$

$$\Rightarrow nx \leq M - x, \forall n \in \mathbb{N}.$$

This shows that $M - x$ is also an upper bound of S and $M - x < M$, a contradiction to the assumption that M is a Sup(S). i. e. $nx < y, \forall n \in \mathbb{N}$ is wrong.

Therefore, $nx > y, \forall n \in \mathbb{N}$.

Corollary 1: If $y \in \mathbb{R}$ then there exists $n \in \mathbb{N}$ *such that* $y < n$.

Proof: Take $x = 1$ in Archimedean property.

Corollary 2: Let x be a positive real number. Then

(a) There exists $n \in \mathbb{N}$ *such that* $0 < \frac{1}{n} < x$.

(b) There exists $m \in \mathbb{N}$ *such that* $m - 1 \leq x < m$.

Theorem 18: Density Theorem:

If x and y are real numbers such that $x < y$ then there exists a rational number $r \in \mathbb{R}$ such that $x < r < y$.

OR

Between any two distinct real numbers there is a rational number.

OR

The set $\mathbb{Q}$ (set of all rational numbers) is dense in $\mathbb{R}$.

Proof:

Here, $x > 0$.

$\therefore$ $we\ have\ 0 < x < y \Rightarrow y - x > 0$.

Now by Archimedean property, there exists $n \in \mathbb{N}\ such\ that\ n(y - x) > 1$

$i.e. ny - nx > 1$ … (i).

Applying (b) of [Corollary 2] to $nx > 0$.

We have, $m \in \mathbb{N}\ such\ that\ m - 1 \leq nx < m$

$\therefore m \leq nx + 1 < ny\ since\ by\ (i)$.

$\Rightarrow nx < m < ny \Rightarrow x < \frac{m}{n} < y$

$i.e. x < r < y;\ \ r = \frac{m}{n}$, a rational number.

Corollary 3: If x and y are real numbers such that $x < y$ then there exists an irrational number $z \in \mathbb{R}$ such that $x < z < y$.

Proof: Apply density theorem to real numbers $\frac{x}{\sqrt{2}}$ and $\frac{y}{\sqrt{2}}$. We will get a rational number $r \neq 0\ such\ that\ \frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}} \Rightarrow x < \sqrt{2}r < y$

$i.e. x < z < y; z = \sqrt{2}r$ is an irrational number.

Examples:

1. Show that $\sqrt{2}$ is not a rational number. OR Show that there does not exists a rational number $x\ such\ that\ x^2 = 2$.

Solution: We will prove this by Contradiction. Let if possible assume that there exists a rational number $\frac{p}{q}\ such\ that\ \left(\frac{p}{q}\right)^2 = 2$.

As $\frac{p}{q}$ is a rational number, we have $p, q \in \mathbb{Z}, q \neq 0\ and\ (p, q) = 1$.

Now, $\left(\frac{p}{q}\right)^2 = 2 \Rightarrow p^2 = 2q^2$ … (a). Here, R. H. S. is an even number

∴ L. H. S. is also an even number. i. e. p^2 is even⇒ p is also even.

Hence, take $p = 2m, for\ some\ m \in \mathbb{Z}$ …. (*)

∴ $p^2 = 2m^2$ … (b). From (a) and (b) we have,

$2q^2 = 4m^2 \Rightarrow q^2 = 2m^2$ which is even. Therefore, q^2 is even

$\Rightarrow q$ is also even number. i. e. $q = 2n, for\ some\ n \in \mathbb{Z}$ …. (**).

Now from (*) and (**) we see that $p\ and\ q$ both have 2 as a common factor. i. e. $(p, q) = 2$, which is a contradiction to our assumption that $(p, q) = 1$. Thus, there does not exists a rational number whose square is 2.

2. Prove that $(i)\sqrt{21}$ is not rational numbers.

Solution: suppose $\sqrt{21}$ is a rational number.

$\therefore \frac{a}{b} = \sqrt{21}, a, b \in \mathbb{Z}\ b \neq 0\ and\ (a, b) = 1$.

$\frac{a^2}{b^2} = 21 \Rightarrow a^2 = 21.b^2 \Rightarrow 21\ divides\ a^2$.

Now, the factors of 21 are 1, 3, 7. Let us consider a factor 3 of 21.

As 3 divides $21b^2$ ∴ 3 divides a^2. Therefore, 3 divides a also.

$\Rightarrow a = 3c, for\ some\ c\ an\ integer$ … (a) $\therefore 21b^2 = 9c^2 \therefore 7b^2 = 3c^2$

$\therefore 3\ divides\ 7b^2 \therefore 3\ divides\ b^2 \Rightarrow 3\ divides\ b.$

$\Rightarrow b = 3d, for\ some\ d\ an\ integer.$ … (b).

From (a) and (b) we see that 3 is a common factor of both a and b.

i.e. (a, b) =3, a contradiction to the fact that (a, b) = 1.

Thus, $\sqrt{21}$ is not a rational number.

3. Prove that $\sqrt{3}+\sqrt{7}$ is not a rational number.

Solution:

On the contrary suppose that $\sqrt{3}+\sqrt{7}$ is a rational number.

$\frac{p}{q}=\sqrt{3}+\sqrt{7}\,;p,q\in\mathbb{Z},q\neq 0\ and\ (p,q)=1.$

$\frac{p^2}{q^2}=\left(\sqrt{3}+\sqrt{7}\right)^2 \Rightarrow p^2=\left(10+2\sqrt{21}\right)q^2=2\left(5+\sqrt{21}\right)q^2$ ..... (i)

This shows that 2 divides $p^2 \Rightarrow 2\ divides\ p. \therefore p=2m, for\ some\ m\in\mathbb{Z}$.

$\therefore p^2=4m^2$.... (ii). From (i) and (ii) we have,

$2\left(5+\sqrt{21}\right)q^2=4m^2 \Rightarrow \left(5+\sqrt{21}\right)q^2=2m^2$

$\Rightarrow 2\ divides\left(5+\sqrt{21}\right)q^2 \Rightarrow 2\ divides\left(5+\sqrt{21}\right)\ or\ q^2$.

$But\ 2\ divides\left(5+\sqrt{21}\right)$ is not possible. $\therefore 2\ divides\ q^2 \Rightarrow 2\ divides\ q$.

$Hence, take\ q=2n, for\ some\ n\in\mathbb{Z}$ .... (iii).

From (ii) and (iii) we see that 2 is a common factor between $p\ and\ q$.

i. e. $(p,q)=2$, a contradiction to $(p,q)=1$.

Thus, $\sqrt{3}+\sqrt{7}$ is not a rational number.

4. Find the rational number $r\ such\ that$

$(i)\ \sqrt{2}<r<\sqrt{3}\ (ii)\ \sqrt{3}<r<\sqrt{5}\ (iii)\ \sqrt{10}<r<\sqrt{11}.$

Solution:

(i) Let $\sqrt{2}<r<\sqrt{3} \therefore 2<r^2<3$. Here, $r^2\ can\ be\ 2.25$
$\Rightarrow r=1.5=\frac{3}{2}$. {Note that this is not unique rational no. between $\sqrt{2}$ and $\sqrt{3}$} $\therefore \sqrt{2}<\frac{3}{2}<\sqrt{3}$.

(ii) Let $\sqrt{3}<r<\sqrt{5} \therefore 3<r^2<5$. Here, $r^2\ can\ be\ 2.56$
$\Rightarrow r=1.6=\frac{5}{3}$. {Note that this is not unique rational no. between $\sqrt{3}$ and $\sqrt{5}$.
$\therefore \sqrt{3}<\frac{5}{3}<\sqrt{5}$.

(iii) Let $\sqrt{10}<r<\sqrt{11} \therefore 10<r^2<11$. Here, $r^2\ can\ be\ 10.24$
$\Rightarrow r=3.2=\frac{16}{5}$. {Note that this is not unique rational no. between $\sqrt{10}$ and $\sqrt{11}$. $\therefore \sqrt{10}<\frac{16}{5}<\sqrt{11}$.

5. Find the Sup. and Inf. of the set $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

Solution: We know that $1 \le n \ \therefore 0 < \frac{1}{n} \le 1, \forall n \in \mathbb{N}$.

Hence, 0 is the lower bound of S and 1 is the upper bound of S.

Therefore, Sup(S) =1 and Inf (S) = 0.

6. Find the Sup. and Inf. of the following sets

(a) $A = \{x \in \mathbb{R}: 2x + 5 > 0\}$

(b) $B = \{x \in \mathbb{R}: x + 2 > x^2\}$.

Solution:

(a) For $x \in \mathbb{R}$, $A = \{x \in \mathbb{R}: 2x + 5 > 0\}$

$$= \left\{x \in \mathbb{R}: x > -\frac{5}{2}\right\}.$$

Therefore, $Inf(A) = -\frac{5}{2}$ and Sup (A) does not exist.

(b) We have, $B = \{x \in \mathbb{R}: x + 2 > x^2\}$

$$= \{x \in \mathbb{R}: x^2 - x - 2 < 0\}$$

$$= \{x \in \mathbb{R}: (x - 2)(x + 1) < 0\}$$

$$= \{x \in \mathbb{R}: -1 < x < 2\}.$$

Therefore, $Inf(B) = -1 \ and \ Sup(B) = 2$.

7. Find the Sup. and Inf. for the sets, if exists:

(i) {-1, 3, 2, 5, 7, 9, 12}. (ii) $\left\{-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\right\}$. (iii) $\left\{\frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$.

(iv) $\left\{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$. (v) $\left\{\left(\frac{1}{2}\right)^n : n \in \mathbb{N}\right\}$. (vi) $\left\{3 + \left(\frac{2}{3}\right)^n : n \in \mathbb{N}\right\}$.

Solution:

(i) Sup = 12, Inf. = -1. (ii) Sup = 0, Inf. = -1.

(iii)Sup = 1/2, Inf. = -1. (iv) Sup = 3/2, Inf. = 0.

(v) Sup =1/2, Inf. = 0. (vi) Sup = 11/3, Inf. = 3.

- **Neighborhood of a Point/ real number:**

If $a \in \mathbb{R}\, and\, \delta > 0$ be a real number. Then δ- nbd of a real number $a;\, denoted\ by\ N_\delta(a) or\ N(a,\delta)$ is defined as-

$$\boldsymbol{N_\delta(a) = N(a,\delta) = \{x \in \mathbb{R} :\ |x - a| < \delta\}}.$$

$i.e. x \in N_\delta(a) \Leftrightarrow |x - a| < \delta$

$$\Leftrightarrow -\delta < x - a < \delta$$

$$\Leftrightarrow a - \delta < x < a + \delta$$

$$\Leftrightarrow (a - \delta, a + \delta).$$

- **Deleted Neighborhood of a point/ real number:**

Let $a \in \mathbb{R}\, and\, \delta > 0$ be a real number. Then δ- nbd of a real number $a;\, denoted\ by\ N'_\delta(a) or\ N'(a,\delta)$ is defined as-

$$\boldsymbol{N'_\delta(a) = N'(a,\delta) = \{x \in \mathbb{R} :\ |x - a| < \delta, x \neq a\}}.$$

$i.e. x \in N'_\delta(a) \Leftrightarrow |x - a| < \delta, x \neq a$

$$\Leftrightarrow -\delta < x - a < \delta, x \neq a$$

$$\Leftrightarrow a - \delta < x < a + \delta, x \neq a$$

$$\Leftrightarrow (a - \delta, a) \cup (a, a + \delta).$$

Illustration: Find $N_\delta(a)\ and\ N'_\delta(a), if\ a = -3\ and\ \delta = 2$.

Solution:

Here, given that $a = -3\ and\ \delta = 2$.

$N_\delta(a) = N(a,\delta) = \{x \in \mathbb{R} :\ |x - a| < \delta\}$

$\therefore N_2(-3) = N(-3, 2) = \{x \in \mathbb{R} :\ |x - (-3)| < 2\}$

$$= -2 < x - (-3) < 2$$

$$= -5 < x < -1$$

$$= (-5, -1).$$

$N'_\delta(a) = N'(a,\delta) = \{x \in \mathbb{R} : |x-a| < \delta, x \neq a\}$.

$\therefore N'_2(-3) = N'(-3,2) = \{x \in \mathbb{R} : |x-(-3)| < 2, x \neq -3\}$

$= -2 < x-(-3) < 2$

$= -5 < x < -1$

$= (-5,-3) \cup (-3,-1)$.

Exercise

1. Draw the graph of the function $f(x) = |x-2|$
2. If x and y are two real numbers then prove that $|x+y| \leq |x|+|y|$.
3. Draw the graph of the function $f(x) = (x+1)^2$. State the intervals in which it is increasing and decreasing.
4. State the order axioms of real numbers.
5. Find the rational number between $(i)\ \sqrt{5}\ \&\ \sqrt{6}\ \ (ii)\ \sqrt{6}\ \&\ \sqrt{7}$
6. For any two distinct, positive real numbers a and b, prove that $\sqrt{ab} < \frac{1}{2}(a+b)$.
7. Solve $\left|\frac{3-x}{2+x}\right| < 1\ \forall x \in \mathbb{R},\ x \neq -2$.
8. State the field axioms of set of real numbers.
9. Find all real numbers x that satisfy the inequality $\left|\frac{2-x}{3+x}\right| < 1,\ x \neq -3$.
10. Determine the set $A = \{x \in \mathbb{R}: |x-1| < 0.5\}$.
11. Find all real numbers x that satisfy the inequality $|x^2-1| \leq 3$.
12. State the completeness property of $\mathbb{R}$.
13. Determine the set $A = \{x \in \mathbb{R}: |2x-3| < 5\}$.
14. If $x \in \mathbb{R}$ then show that there exists $n_0 \in \mathbb{N}$ such that $x < n_0$.
15. Show that between any two distinct real numbers there exists a rational number.
16. Find l. u. b. and g. l. b. for the set $S = \left\{\frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$.
17. Find Infimum and Supremum of the set $S = \left\{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$.
18. Determine the set $A = \{x \in \mathbb{R} : x^2 > 3x+4\}$.
19. State order axioms for set of real numbers.
20. Prove that for $x, y \in \mathbb{R}$, $\big||x|-|y|\big| \leq |x-y|$.
21. Find all real numbers x that satisfy the inequality
    $|x|+|x+1| < 2$.
22. Solve the inequality $|3x+4| < |x+2|$
23. Find the greatest lower bound and least upper bound of the set
    $S = \left\{(-1)^n + \frac{n+1}{n+2} : n \in \mathbb{N}\right\}$.

24. Find the greatest lower bound and least upper bound of the set

$$S = \left\{-\frac{1}{3}, \frac{7}{4}, -\frac{1}{5}, \frac{11}{6}, -\frac{1}{7}, \dots\right\}.$$

25. Solve the inequality $4 - 7x < 3x - 16$.
26. Find the Supremum and Infimum of the set $S = \left\{\frac{n-1}{n} : n \in \mathbb{N}\right\}$.
27. If x is positive real number then prove that for any real number y there exists a natural number n such that $nx > y$.
28. Find the Supremum and Infimum of the set $S = \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$.
29. Define absolute value of areal number. If $a \geq 0$ then prove that $|x| \leq a$ if and only if $-a \leq x \leq a$.
30. For x and y any two real numbers, prove that $|x + y| \leq |x| + |y|$. Hence prove that $|x - y| \leq |x| + |y|$.
31. Determine the set $A = \{x \in \mathbb{R} : 12x + 3 < 7\}$.
32. Find the Supremum and Infimum of the set A, if exist, where $A = \{-1, 3, 2, 0, 9, 12\}$.
33. Find the domain and range of the function $y = \sqrt{25 - x^2}$.
34. Sketch the graph of the function $f(x) = x^2, x \in [-1, 1]$.
35. For all a, b in $\mathbb{R}$ prove that $|a + b| \leq |a| + |b|$.
36. Find all $x \in \mathbb{R}$ that satisfy the inequality $|4x + 5| \leq 19$.
37. Find Supremum and Infimum of the set $\left\{\left(\frac{1}{2}\right)^n : n \in \mathbb{N}\right\}$.
38. If $c \in \mathbb{R}\ and\ 0 < c < 1$ then show that $0 < c^2 < c < 1$.
39. State the density theorem.
40. Find the range of the function $f(x) = x^2 + 1, x \in \mathbb{R}$.
41. Find all real numbers that satisfies the inequality $|x - 1| < |x|$.
42. Find the Supremum and Infimum of the set $\{x \in \mathbb{R} : x + 2 > x^2\}$.
43. Determine the set $A = \left\{x \in \mathbb{R} : x < \frac{1}{x}, x > 0\right\}$.
44. Find the domain of the function $(x) = \frac{1}{x-3}$.
45. State and prove the triangle inequality for real numbers. Hence prove that $|a - b| \geq \big||a| + |b|\big|$, for all $a, b \in \mathbb{R}$.
46. Draw the graph of the function $f(x) = 3x^2 - 7$.

-$$$$$$$$ -

Unit 4: Limits and Continuity

Introduction:

In this chapter, we are going to have revision of function, domain and range of a function along with some of the example. Define cluster point, deleted neighborhood of a point, the limit of a function of one variable. Examples of function, properties, theorems. Define the continuity of a function at a point and on an interval. Some theorems and examples on continuity.

Definition: Function- A function from a set A to set B is a relation which associates to every element in set A unique element in set B. It is denoted by $f: A \to B$.

The set A is called **domain** and the second set B is called **co-domain** of the function.

More clearly, domain of a function is the set of all values of the variable in Set A for which the function will be well defined.

Range of a function: If $f: A \to B$ is a function then the set denoted by $R(f)$; defined as $R(f) = \{f(x): \forall\, x \in A\}$ is called range of function f.

Note that the domain assumed to be a subset of set of Real numbers which is called as Natural domain.

Examples:

Find the natural domain for each of the following function

$(i)\ f(x) = x^2 + 2 \quad (ii)\ f(x) = \frac{1}{x-3} \quad (iii)\ g(x) = \sqrt{1+6x} \quad (iv)\ h(x) = \sqrt{1-x^2}$

Solution: (i) Here, given function $f(x) = x^2 + 2$ is well defined for all values of x to be real numbers. Therefore, the domain of function is set $\mathbb{R}$ i. e. $(-\infty, \infty)$.

(ii) Here, the given function $f(x) = \frac{1}{x-3}$ is not defined at $x = 3$.

Therefore, $D(f) = \{\, x \in \mathbb{R}: x \neq 3\}$.

(iii) In this case for $g(x) = \sqrt{1+6x}$, we know that square root exist if and only if the quantity under square root is non – negative.

$$D(g) = \{x \in \mathbb{R}: 1 + 6x \geq 0\} = \left\{x \in \mathbb{R}: x \geq -\frac{1}{6}\right\} = \left[-\frac{1}{6}, \infty\right).$$

(iv) Here also,

$$D(h) = \{x \in \mathbb{R}: 1 - x^2 \geq 0\} = \{x \in \mathbb{R}: 1 \geq x^2 \geq 0,\ \}.$$

$$= \{x \in \mathbb{R}: -1 \leq x \ \leq 1\,\} = [-1, 1]$$

- **Rule to find the range of a function:**

Step 1: Put $y = f(x)$.

Step 2: Solve the relation in x and y for x, instead of y.

Step 3: The range is the set of all real numbers y that can be solved for x.

Examples:

1. For the following function, find the range

$(i)\ f(x) = x^2 + 2\ (ii) f(x) = \frac{1}{x-3}\ (iii) g(x) = \sqrt{1+6x}\ (iv)\ h(x) = \sqrt{1-x^2}$

Solution:

(i) Put $y = f(x) = x^2 + 2$

$$\Rightarrow x^2 = y - 2 \therefore x = \pm\sqrt{y-2}\,.$$

Hence, $R(f) = \{y \in \mathbb{R}: y - 2 \geq 0\} = [2, \infty)$.

(ii) Let $y = f(x) = \frac{1}{x-3} \Rightarrow x = \frac{1}{y} + 3$.

Thus, $R(f) = \{\, y \in \mathbb{R}: y \neq 0\} = \mathbb{R} - \{0\}$.

(iii) Let $y = g(x) = \sqrt{1+6x}$

$$\Rightarrow y^2 = 1 + 6x \ \therefore x = \frac{y^2-1}{6}, y^2 \geq 0.$$

Therefore, $R(g) = \{y \in \mathbb{R}: y^2 \geq 0\} = (-\infty, \infty)$.

(iv) Let $y = h(x) = \sqrt{1-x^2}$

$$\Rightarrow y^2 = 1 - x^2, y \geq 0$$

$$\therefore x^2 = 1 - y^2 \therefore x = \pm\sqrt{1-y^2}, y \geq 0.$$

Hence, $R(h) = \{y \in \mathbb{R}: y^2 \leq 1\} = [0, 1]$.

2. Find the domain of the function$f(x) = \sqrt{x+7} - \sqrt{x^2+2x-15}$.

Solution: We have $f(x) = \sqrt{x+7} - \sqrt{x^2+2x-15}$.

Here, it is defined iff $x+7 \geq 0$ and $x^2+2x-15 \geq 0$.

$\therefore x \geq -7$ and $(x+5)(x-3) \geq 0$.

(i) $x \geq -7$ and $(x+5) \geq 0, (x-3) \geq 0$

(ii) $x \geq -7$ and $(x+5) \leq 0, (x-3) \leq 0$. $\Rightarrow x \geq -7$ and $x \geq 3$ or $x \leq -5$.

Thus, $D(f) = \{ x \in \mathbb{R}: x \geq -7 \text{ and } (x \geq 3 \text{ or } x \leq -5)\}$

$= \{x \in \mathbb{R}: x \geq -7 \text{ or } -7 \leq x \leq -5\}$

$= [3, \infty) \cup [-7, -5]$.

3. Find the range of the function $f(x) = \frac{2x+1}{x^2+1}$.

Solution: Let $y = f(x) = \frac{2x+1}{x^2+1}$ $\therefore yx^2 + y = 2x + 1$

$\therefore yx^2 - 2x + (y - x) = 0$; Which is a quadratic equation in x with 'y' being coefficient.

$x = \frac{\left(2 \pm \sqrt{4-4y(y-1)}\right)}{2y}$ $, y \neq 0$ and $x = \frac{\left(1 \pm \sqrt{1-y^2+y}\right)}{y}$ $, y \neq 0$.

This can be solved if and only if $1 - y^2 + y \geq 0$

$\Rightarrow y^2 - y + 1 \leq 0 \Rightarrow \left(y - \frac{1-\sqrt{5}}{2}\right)\left(y - \frac{1+\sqrt{5}}{2}\right) \leq 0$.

Thus, $R(f) = \{y \in \mathbb{R}: y^2 - y - 1 \leq 0\} = \left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$.

Absolute Value function:

The absolute value function $f: \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0. \end{cases}$$

The number $|a|$ is called absolute value of a. The range of this function is $[0, \infty)$.

Piecewise Function: Piecewise functions are common in mathematics, physics, and engineering to model situations where a process or relationship changes over different intervals.

Step function: It is a piecewise function where it has jumps from one value to another value. i. e. A stepwise function, also known as a step function, is a piecewise function that remains constant within each interval of its domain.

Increasing and decreasing function:

Let f be a function defined on I = [a, b]. Let $x_1, x_2 \in [a, b]$

(i) f is said to be increasing on I if $f(x_1) > f(x_2), x_1 > x_2$.

(ii) f is said to be decreasing on I if $f(x_1) < f(x_2), x_1 > x_2$.

Even and Odd function:

A function $y = f(x)$ is called Even function if $f(x) = f(-x)$ and Odd function if $f(-x) = -f(x)$, for every x in domain of the function.

- Note that the graph of the Even function is symmetric about the y – axis and the graph of the Odd function is symmetric about the x – axis.
- The sum or difference of two even functions is even. The product of two even functions is even.
- The sum or difference of two odd functions is odd. The product of two odd functions is even, while the product of an even function and an odd function is odd.
- A function can be both even and odd only if it is the zero function, f(x)=0 for all x
- A function can be neither even nor odd. For example, $f(x) = x + 1$ does not satisfy the conditions for either evenness or oddness.

Limit of a function

Definition: Cluster Point: -

Let $A \subseteq \mathbb{R}$. A point c $\in \mathbb{R}$ is a cluster point of A if for every $\delta > 0$ there exist at least one point $x \in A, x \neq c \ s.t. |x - c| < \delta$.

- A point c is said to be cluster point of the set A if every deleted δ- neighborhood $N'_\delta(c) = (c - \delta, c + \delta)$ of c contains at least one point of A. i. e. $N'_\delta(c) \cap A \neq \emptyset$.

Remark:

- The point c may or may not be in A.
- If a function is discontinuous at a cluster point, it means the limit does not exist, or it does not equal the function's value at that point.
- Cluster points, also known as limit points or accumulation points, play an important role in the study of limits and continuity in real analysis and topology.

Examples:

1. The set of natural numbers has no cluster points.

Solution: The set of natural numbers has no cluster points because there is no real number around which the natural numbers accumulate; each natural number is isolated with gaps between them. Consequently, no point has natural numbers arbitrarily close to it.

2. Finite set has no cluster points.

Solution: A finite set has no cluster points because there are only a limited number of isolated points, with no point having other points arbitrarily close to it. Cluster points require an infinite accumulation of points around them.

3. Let A = {1, 2} then A has no limit points.

Solution: Take $\delta = \frac{1}{2}, c = 1.\ Then\ N'_\delta(c) = (c-\delta, c+\delta) - \{c\} = \left(\frac{1}{2}, \frac{3}{2}\right) - \{1\}$.

$$\therefore N'_\delta(c) \cap A = \left[\left(\frac{1}{2}, \frac{3}{2}\right) - \{1\}\right] \cap \{1,2\} = \emptyset.$$

Therefore, 1 is not limit point of A. Similarly, 2 is also not limit point of A.

4. Let A = (1, 2) then 1 and 2 are limit points of A.

Solution: Take $\delta = \frac{1}{2}, c = 1. Then\ N'_\delta(c) = (c-\delta, c+\delta) - \{c\}$

$$= \left(\frac{1}{2}, \frac{3}{2}\right) - \{1\}.$$

$$\therefore N'_\delta(c) \cap A = \left[\left(\frac{1}{2}, \frac{3}{2}\right) - \{1\}\right] \cap (1,2) \neq \emptyset.$$

Therefore, 1 is a limit point of A. Similarly, 2 is limit point of A.

Here, every point of A is a limit point of A. Except the end points 1 and 2 are the limit points of A which does not in A.

Deleted neighborhood of a point- Let $c \in \mathbb{R}\ and\ \delta > 0$ be any positive real number. Then deleted neighborhood of a point c, denoted by $N'_\delta(c) or\ N'(c,\delta)$; is defined as

$$N'_\delta(c) = N'(c,\delta) = \{x \in \mathbb{R}: 0 < |x-c| < \delta\}$$

$$= \{x \in \mathbb{R}: c-\delta < x < c+\delta, x \neq c\}$$

$$= \{x \in \mathbb{R}: c-\delta < x < c\ or\ c < x < c+\delta\}$$

$$= \{x \in \mathbb{R}: c-\delta < x < c\} \cup \{x \in \mathbb{R}: c < x < c+\delta\}$$

$$= (c-\delta, c+\delta) - \{c\}.$$

Example: Find deleted 3 nbd. of 1.

Solution: Here, $\delta = 3 \; and \; c = 1$.

$$\therefore N'(c,\delta) = (c-\delta, c+\delta) - \{c\} = (1-3,1) \cup (1,1+3)$$

$$= (-2,1) \cup (1,4)$$

$$= (-2,4) - \{1\}.$$

Definition: Limit of a function-

Let $f(x)$ be a function defined on A⊆ℝ and let c be a cluster point of A. A number $l \in \mathbb{R}$ is called a limit of function $f(x)$ at $x = c$ if for given $\varepsilon > 0 \; \exists \; \delta > 0 \; s.t.$

$$x \in A, 0 < |x-c| < \delta \Rightarrow |f(x) - l| < \varepsilon.$$

$We \; will \; write \; this \; as \; \lim_{x \to c} f(x) = l.$

Remark:

1. The value of δ depends on ε.

2. The inequality $0 < |x-c|$is equivalent to saying that $x \neq c$.

3. $\lim_{x \to c} f(x) = l$ is equivalent to

   (a) $f(x)$ Approaches to l as x approaches to c.

   (b) $f(x) \to l \; \to as \; x \to c$.

4. If limit of $f \; at \; x = c$ does not exists then we say that $f(x)$ diverges at $x = c$.

Theorem 1: If $f: A \to \mathbb{R}$ and c is a cluster point of A then $f(x)$ has only one limit at c.

OR

Show that limit of a function is unique if it exists.

Proof: Let if possible $l_1 and \; l_2$ be two limits of a function $f(x) \; at \; x = c$.

By the definition of limit, for given $\varepsilon > 0, \exists \; \delta_1, \delta_2 > 0 \; s.t.$

$$x \in A, 0 < |x-c| < \delta_1 \Rightarrow |f(x) - l_1| < \frac{\varepsilon}{2} \text{ .... (1) and}$$

$$x \in A, 0 < |x-c| < \delta_2 \Rightarrow |f(x) - l_2| < \frac{\varepsilon}{2} \text{ .... (2).}$$

Let $\delta = \min\{\delta_1, \delta_2\}$ then both eqn (1) and eqn (2) holds whenever $0 < |x-c| < \delta$.

Claim: we need to show that $l_1 = l_2$

For consider,

$$|l_1 - l_2| = |(f(x) - l_1) - (f(x) - l_2)| \leq |f(x) - l_1| + |f(x) - l_2|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, whenever $0 < |x - c| < \delta$ *and* $x \in A$, *we have* $|l_1 - l_2| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this is not possible. Therefore, we have $l_1 - l_2 = 0 \Rightarrow l_1 = l_2$. This shows that limit of a function if exist is unique.

Examples: Using the definition, show that

(i) $\lim_{x\to 1}(x^2 + 1) = 2$. (ii) $\lim_{x\to -1}\left(\frac{x+5}{2x+3}\right) = 4$

(iii) $\lim_{x\to 1}\frac{x}{1+x} = \frac{1}{2}$ (iv) $\lim_{x\to 2}(x^2 + 4x) = 12$.

Solution:

(i) Let $f(x) = x^2 + 1$, *here* $l = 2, c = 1$.

We want to make $|f(x) - 2| < \varepsilon$ by taking x sufficiently close to c = 1.

Consider the deleted δ-nbd of c = 1. (Take $\delta = 1$).

Say $S = (c - \delta, c + \delta) - \{1\} = (0, 2) - \{1\}$.

Here, $|x - c| < \delta$ *means* $|x - 1| < 1$……….. (1)

Consider,

$$|f(x) - 2| = |x^2 + 1 - 2| = |x^2 - 1| = |x + 1||x - 1| \quad ……… *$$

Now, in S, $x > 0 \Rightarrow x + 1 < 3 \Rightarrow |x + 1| < 3$……….. (2)

With this eqn * becomes, $|f(x) - 2| < 3|x - 1|$.

But we need $3|x - 1| < \varepsilon \Rightarrow |x - 1| < \frac{\varepsilon}{3}$………… (3).

From eqn (1) and eqn (3), we will choose $\delta = \min\left\{1, \frac{\varepsilon}{3}\right\}$.

Then $|f(x) - 2| < \varepsilon$, *whenever* $0 < |x - 1| < \delta \Rightarrow \lim_{x\to 1}(x^2 + 1) = 2$.

(ii) Let $f(x) = \frac{x+5}{2x+3}$, $here\ \ l = 4\,, c = -1.$

We want to make $|f(x) - 4| < \varepsilon$ by taking x sufficiently close to c = -1.

Consider the deleted δ-nbd of c = -1. (Take $\delta = 1$).

$$\text{Say } S = (c-\delta, c+\delta) - \{-1\} = (-2, 0) - \{-1\}.$$

$Here,\ |x - c| < \delta\ means\ |x-(-1)| < 1$.... (1)

Consider,

$$|f(x) - 4| = \left|\frac{x+5}{2x+3} - 4\right| = \left|\frac{x+5-4(2x+3)}{2x+3}\right| = \left|\frac{-7x-7}{2x+3}\right| = \frac{7}{2}\cdot\frac{|x+1|}{\left|x+\frac{3}{2}\right|} \text{ .... } *$$

Now, in S, $x > -2 \Rightarrow 2x+3 > 1 \Rightarrow \frac{1}{2x+3} < 1\ \ \therefore \frac{1}{\left|x+\frac{3}{2}\right|} < 4$ .... (2)

With this eqn * becomes, $|f(x) - 4| < \frac{7}{2}.4|x+1| = 14|x+1|$.

$$But\ we\ need\quad 14|x+1| < \varepsilon \Rightarrow |x+1| < \frac{\varepsilon}{14} \text{.......... (3).}$$

From eqn (1) and eqn (3), we will choose $\delta = \min\left\{1, \frac{\varepsilon}{14}\right\}$.

Then $|f(x) - 4| < \varepsilon, whenever\ 0 < |x+1| < \delta$

$$\Rightarrow \lim_{x\to -1}\frac{x+5}{2x+3} = 4.$$

(iii) Let $f(x) = \frac{x}{1+x}$, $here\ \ l = \frac{1}{2}\,, c = 1.$

We want to make $\left|f(x) - \frac{1}{2}\right| < \varepsilon$ by taking x sufficiently close to c = 1.

Consider the deleted δ-nbd of c = 1. (Take $\delta = 1$).

Say $S = (c-\delta, c+\delta) - \{1\} = (0,2) - \{1\}$.

$Here,\ |x - c| < \delta\ means\ |x - 1| < 1$.......... (1)

Consider,

$$\left|f(x) - \frac{1}{2}\right| = \left|\frac{x}{1+x} - \frac{1}{2}\right| = \left|\frac{2x-1-x}{2(1+x)}\right| = \left|\frac{x-1}{2(x+1)}\right| = \frac{|x-1|}{2|x+1|} \text{ .... } *$$

Now, in S, $x > 0 \Rightarrow x+1 > 1 \Rightarrow \frac{1}{x+1} < 1.\ \ \therefore \frac{1}{|x+1|} < 1$ .... (2)

With this eqn * becomes, $\left|f(x) - \frac{1}{2}\right| < \frac{|x-1|}{2}$.

$$But\ we\ need\ \frac{|x-1|}{2} < 1 \Rightarrow |x-1| < 2\varepsilon \text{.......... (3).}$$

From eqn (1) and eqn (3), we will choose $\delta = \min\{1, 2\varepsilon\}$.

Then $\left|f(x) - \frac{1}{2}\right| < \varepsilon, whenever\ 0 < |x-1| < \delta \Rightarrow \lim_{x\to 1} \frac{x}{1+x} = \frac{1}{2}$.

(iv) Let $f(x) = x^2 + 4x,\ \ here\ \ l = 12\,, c = 2$.

We want to make $|f(x) - 12| < \varepsilon$ by taking x sufficiently close to c = 2.

Consider the deleted δ-nbd of c = 1. (Take $\delta = 1$).

$$\text{Say } S = (c-\delta, c+\delta) - \{2\} = (1,3) - \{2\}.$$

$Here,\ |x-c| < \delta\ means\ |x-2| < 1$……… (1)

Consider,

$$|f(x) - 12| = |x^2 + 4x - 12| = |x+6||x-2| \ \text{……….} *$$

Now, in S, $x > 1 \Rightarrow x + 6 = 9 \Rightarrow |x+6| < 9$ ………. (2)

With this eqn * becomes, $|f(x) - 12| < 9|x-2|$.

$$But\ we\ need\quad 9|x-2| < \varepsilon \Rightarrow |x-2| < \frac{\varepsilon}{9} \text{………… (3).}$$

From eqn (1) and eqn (3), we will choose $\delta = \min\left\{1,\ \frac{\varepsilon}{9}\right\}$.

Then $|f(x) - 12| < \varepsilon, whenever\ 0 < |x-2| < \delta\ \Rightarrow \lim_{x\to 2}(x^2 + 4x) = 12$.

Examples: Prove that

(i) $\lim_{x\to 1}(x^2 + 4x) = 5$. $(ii)\ \lim_{x\to -2}(x^2 + 3x) = -2$. $(iii)\ \lim_{x\to 3}(x^2 + 2x) = 15$.

$(iv)\ \lim_{x\to 0}\left(\frac{-9x^2+4}{3x+2}\right) = 2$. $(v)\ \lim_{x\to 0}\left(\frac{2x^2+3}{x+5}\right) = \frac{3}{5}$.

Solution: (iv) Let $f(x) = \left(\frac{-9x^2+3x}{3x+2}\right),\ \ here\ \ l = 2\,, c = 0$.

We want to make $|f(x) - 2| < \varepsilon$ by taking x sufficiently close to c = 0.

Consider the deleted δ-nbd of c = 0. (Take $\delta = 1$).

$$\text{Say } S = (c-\delta, c+\delta) - \{0\} = (-1,1) - \{0\}.$$

$Here,\ |x-c| < \delta\ means\ |x-0| < 1$……….. (1)

Consider,

$$|f(x) - 2| = \left|\left(\frac{-9x^2+4}{3x+2}\right) - 2\right| = \left|\left(-\frac{3x(3x+2)}{3x+2}\right)\right| = |-3x| = 3|x| \ \text{……….} *$$

Now, in S, $x > -1 \Rightarrow 3|x| < 1$, ………… (2)

With this eqn* becomes, $|f(x) - 2| < 3|x|$.

$$But\ we\ need \quad 3|x| < \varepsilon \Rightarrow |x| < \frac{\varepsilon}{3} \text{……….. (3).}$$

From eqn (1) and eqn (3), we will choose $\delta = \min\left\{1,\ \frac{\varepsilon}{3}\right\}$.

Then $|f(x) - 2| < \varepsilon, whenever\ 0 < |x - 0| < \delta$

$$\Rightarrow \lim_{x\to 0}\left(\frac{-9x^2+4}{(3x+2)}\right) = 2.$$

(v) Let $f(x) = \frac{2x^2+3}{x+5},\ \ here\ \ l = \frac{3}{5}\ , c = 0.$

We want to make $\left|f(x) - \frac{3}{5}\right| < \varepsilon$ by taking x sufficiently close to c = 0.

Consider the deleted δ-nbd of c = 0. (Take $\delta = 1$).

$$\text{Say } S = (c - \delta, c + \delta) - \{0\} = (-1, 1) - \{0\}.$$

$Here,\ |x - c| < \delta\ means\ |x - 0| < 1$……….. (1)

Consider,

$$\left|f(x) - \frac{3}{5}\right| = \left|\frac{2x^2+3}{(x+5)} - \frac{3}{5}\right| = \left|\frac{10x^2-3x}{5(x+5)}\right| = \left|\frac{10x-3}{5(x+5)}\right||x| \le \left|\frac{10x+50}{5(x+5)}\right||x|$$

$$< 2|x| \text{ ………….. *}$$

$$But\ we\ need \quad 2|x| < \varepsilon \Rightarrow |x| < \frac{\varepsilon}{2} \text{…………. (3).}$$

From eqn (1) and eqn (3), we will choose $\delta = \min\left\{1, \frac{\varepsilon}{2}\right\}$.

Then $\left|f(x) - \frac{3}{5}\right| < \varepsilon, whenever\ 0 < |x - 0| < \delta$

$$\Rightarrow \lim_{x\to 0}\left(\frac{2x^2+3}{x+5}\right) = \frac{3}{5}.$$

Definition: Let $A \subseteq \mathbb{R}, f: A \to \mathbb{R}$ be the function and c be the cluster point of A. We say that the function f is bounded on neighborhood of c if there exists a δ- nbd of c and a constant $M > 0\ s.t.$

$$|f(x)| \le M, \forall\ x \in A \cap N_\delta(c).$$

Theorem 2: If $A \subseteq \mathbb{R}\; anf\; f: A \to \mathbb{R}$ has limit point at x = c in set of real nos. then f is bounded on some nbd of c.

Proof: Let $\lim_{x \to c} f(x) = L$.

By definition; $for\ given\ \varepsilon > 0, \exists\ \delta > 0\ s.t. 0 < |x - c| < \delta$

$\Rightarrow |f(x) - L| < \varepsilon.$

$$Consider, |f(x)| - |L| \leq |f(x) - L| < \varepsilon, whenever\ 0 < |x - c| < \delta.$$

$\therefore 0 < |x - c| < \delta \Rightarrow |f(x)| - |L| < \varepsilon.$

$\Rightarrow |f(x)| < |l| + \varepsilon \Rightarrow |f(x)| < M; M = |l| + \varepsilon.$

Therefore, $f(x)$ is bounded on $N_\delta(c)$. Take $M = Sup\{f(c), |L| + \varepsilon\}$.

Then if $x \in A \cap N_\delta(c) \Rightarrow |f(x)| \leq M.$ i. e. f is bounded on $N_\delta(c)$.

Theorem 3: Algebra of limits-

Let $A \subseteq \mathbb{R}\; and\; f, g$ be functions on A to $\mathbb{R}$. If $cc \in \mathbb{R}$ is the cluster point of A and $\lim_{x \to c} f(x) = l, \lim_{x \to c} g(x) = m$ then

(i) $\lim_{x \to c}(f(x) \pm g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = l \pm m.$

(ii) $\lim_{x \to c}\big(f(x). g(x)\big) = \lim_{x \to c} f(x) . \lim_{x \to c} g(x) = l. m$.

(iii) $\lim_{x \to c}\big(k. f(x)\big) = k. \lim_{x \to c} f(x) = k. l.$

(iv) $\lim_{x \to c}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{l}{m}$ $, m \neq 0, g(x) \neq 0, \forall\ x \in A.$

Theorem 4: Let $A \subseteq \mathbb{R}, f: A \to \mathbb{R}$ be a function and $c \in \mathbb{R}$ be a cluster point of A. If $a \leq f(x) \leq b, \forall\ x \in A, x \neq c\ and\ if$ $\lim_{x \to c} f(x)$ exists then $a \leq \lim_{x \to c} f(x) \leq b.$

Theorem 5: (Squeeze Theorem) If $A \subseteq \mathbb{R}, f, g, h: A \to \mathbb{R}$ be a function and $c \in \mathbb{R}$ be a cluster point of A and If $f(x) \leq g(x) \leq h(x), \forall\ x \in A, x \neq c\ with$ $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$ then $\lim_{x \to c} g(x) = L.$

Definition: Left hand and Right hand limit-

(i) Let $f(x)$ be a function defined on A⊆ℝ and let c be a cluster point of A. A number $l \in \mathbb{R}$ is called a Right hand limit of function $f(x)$ at $x = c$ if for given $\varepsilon > 0 \, \exists \, \delta > 0 \; s.t. \, x \in A, c < x < c + \delta$

$\Rightarrow |f(x) - l| < \varepsilon$. We can denote this as $\lim\limits_{x \to c^+} f(x) = L$.

(ii) Let $f(x)$ be a function defined on A⊆ℝ and let c be a cluster point of A. A number $l \in \mathbb{R}$ is called a left hand limit of function $f(x)$ at $x = c$ if for given $\varepsilon > 0 \, \exists \, \delta > 0 \; s.t. \, x \in A, c - \delta < x < c$

$\Rightarrow |f(x) - l| < \varepsilon.$ We can denote this as $\lim\limits_{x \to c^-} f(x) = L$.

Theorem 6: Let $A \subseteq \mathbb{R}, f: A \to \mathbb{R}$ be a function and $c \in \mathbb{R}$ be a cluster point of A. If $\lim\limits_{x \to c} f(x)$ exists if and only if $\lim\limits_{x \to c^-} f(x) \; and \; \lim\limits_{x \to c^+} f(x) \; exists \; and \; \lim\limits_{x \to c^-} f(x) = \lim\limits_{x \to c^+} f(x)$. We have $\lim\limits_{x \to c^-} f(x) = \lim\limits_{x \to c} f(x) = \lim\limits_{x \to c^+} f(x)$.

Remark:

1. Left hand and Right hand limit are called one sided limits of a function at a point.

2. It may possible that one of them may exist or both may exist and are different.

Infinite limits:

Let $A \subseteq \mathbb{R}, f: A \to \mathbb{R}$ be a function and $c \in \mathbb{R}$ be a cluster point of A. (i) We say that $f \to \infty \; as \; x \to c$ if for every $\alpha \in \mathbb{R} \, \exists \, \delta > 0 \; s.t. \, x \in A. \, 0 < |x - c| < \delta \Rightarrow f(x) > \alpha$.

We can write this as $\lim\limits_{x \to c} f(x) = \infty$.

(ii)We say that $f \to \infty \; as \; x \to c$ if for every $\beta \in \mathbb{R} \, \exists \, \delta > 0 \; s.t.$

$$x \in A, 0 < |x - c| < \delta \Rightarrow f(x) < \beta.$$

$This \; can \; be \; written \; as \; \lim\limits_{x \to c} f(x) = \infty$.

Illustrative Example:

Let $f(x) = x^2 - 2, c = 3 \; then \; \lim\limits_{x \to 3} (x^2 - 2) = 7$.

For, $x > 3 \; i.e. \, x \to 3^+$.

If $x = 3.01 \; then \; f(x) = 7.0601$,

$x = 3.001 \; then \; f(x) = 7.006001$,

$x = 3.0001\ then\ f(x) = 7.00060001,$

$x = 3.00001\ then\ f(x) = 7.0000600001.\ and$

For, $x < 3\ \ i.e. x \to 3^-$.

If $x = 2.99\ then\ f(x) = 6.9401,$

$x = 2.999\ then\ f(x) = 6.994001,$

$x = 2.9999\ then\ f(x) = 6.99940001,$

$x = 2.99999\ then\ f(x) = 6.9999400001.$

Here, in both the cases f(x) is very close to 7.

Hence, $\lim_{x\to 3}(x^2 - 2) = 7$.

Example: 1. Find the right hand and left hand limit of a function

$$f(x) = \begin{cases} \left(\frac{|x-3|}{x-3}\right), & x \neq 3 \\ 0, & x = 3 \end{cases}$$

Solution: If $x > 3\ then\ \lim_{x\to 3^+} f(x) = \lim_{x\to 4^+}\left(\frac{|x-3|}{x-3}\right)$

$$= \lim_{x\to 3^+}\left(\frac{x-3}{x-3}\right) = 1.$$

If $x < 3\ then\ \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}\left(\frac{|x-3|}{x-3}\right)$

$$= \lim_{x\to 3^-}\left(\frac{-(x-3)}{x-3}\right) = -1.$$

$\therefore\ \lim_{x\to 3^+} f(x) \neq \lim_{x\to 3} f(x)$

$\Rightarrow \lim_{x\to 3} f(x)$ does not exist.

2. Evaluate $\lim_{x\to 0}\left[\frac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}+1}\right]$ if it exist.

Solution: We have if $x > 0\ then \frac{1}{x} \to \infty\ \therefore\ e^{\frac{1}{x}} \to \infty.$

If $x < 0, \frac{1}{x} \to -\infty\ \therefore\ e^{\frac{1}{x}} \to 0.$

$\therefore \lim_{x\to 0^+}\left[\frac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}+1}\right] = \lim_{x\to 0^+}\left[\frac{1}{1+e^{\frac{-1}{x}}}\right] = 1.\ and\ \lim_{x\to 0^-}\left[\frac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}+1}\right] = 0.$

$\Rightarrow \lim_{x\to 0}\left[\frac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}+1}\right]$ does not exist.

3. Evaluate $\lim_{x\to -2}\frac{(x+3)|x+2|}{x+2}$.

Solution: $\lim_{x\to -2^+}\frac{(x+3)|x+2|}{x+2} = \lim_{x\to -2^+}\frac{(x+3)(x+2)}{x+2}$

$$= \lim_{x\to -2^+}(x+3) = 1 \ldots. (1)$$

Again $\lim_{x\to -2^-}\frac{(x+3)|x+2|}{x+2} = \lim_{x\to -2^-}\frac{(x+3).-(x+2)}{x+2}$

$$= \lim_{x\to -2^-}(x+3) = -1 \ldots.(2)$$

From (1) and (2) we see that given limit does not exists.

4. Find the limit of $f(x) = \left[\frac{xe^{\frac{1}{x}}}{e^{\frac{1}{x}}+1}\right], x \neq 0$.

5. Find the limit of $f(x) = \begin{cases} \frac{x^2}{a} - a, & 0 < x < a \\ 0, & x = a \\ a - \frac{a^3}{x^2}, & x > a \end{cases}$

6. Find the limit of $f(x) = \frac{x-|x|}{|x|}, x \neq 0$.

7. Find $\lim_{x\to 0}\left(\frac{3x+|x|}{7x-5|x|}\right), x \neq 0$.

8. Prove that $\lim_{x\to 0}\left(\frac{x^2}{3x+|x|}\right) = 0$ if it exists.

9. Prove that $\lim_{x\to 0}\left(x sin\left(\frac{1}{x}\right)\right) = 0$.

Solution: Since $-1 \leq sin x \leq 1$, for all real nos. x.

$$\Rightarrow -1 \leq \sin\left(\frac{1}{x}\right) \leq 1, x \neq 0. \;\; \therefore \;\; |x| \leq \mathrm{x}\sin\left(\frac{1}{x}\right) \leq |x|,\; x \neq 0.$$

By Squeeze theorem, $\lim_{x\to 0}\left(x sin\left(\frac{1}{x}\right)\right) = 0$.

More Examples:

1. Evaluate $\lim\limits_{x \to 1} \frac{x^3-1}{x^2-1}$.

Solution: $\lim\limits_{x \to 1} \frac{x^3-1}{x^2-1} = \lim\limits_{x \to 1} \left[\frac{(x-1)(x^2+x+1)}{(x-1)(x+1)}\right] = \lim\limits_{x \to 1} \left[\frac{x^2+x+1}{x+1}\right] = \frac{3}{2}$ (By quotient rule)

2. $\lim\limits_{x \to 4} \frac{4-\sqrt{x+12}}{x-4}$

Solution:$\lim\limits_{x \to 4} \frac{4-\sqrt{x+12}}{x-4} = \lim\limits_{x \to 4} \left\{\frac{4-\sqrt{x+12}}{x-4} \times \frac{4+\sqrt{x+12}}{4+\sqrt{x+12}}\right\} = \lim\limits_{x \to 4} \left\{-\frac{1}{4+\sqrt{x+12}}\right\} = -\frac{1}{8}$.

3. Evaluate $\lim\limits_{x \to 0} \frac{x^2-x+sinx}{2x}$.

Solution: $\lim\limits_{x \to 0} \frac{x^2-x+sinx}{2x} = \lim\limits_{x \to 0} \left[\frac{x}{2} - \frac{1}{2} + \frac{sinx}{2x}\right] = 0 + \frac{1}{2} + \frac{1}{2}.1 = 0.$

Examples: Using Squeeze theorem show that

(i) $\lim\limits_{x \to 0} sinx = 0$ (ii) $\lim\limits_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0.$

Solution: (i) Let $f(x) = sinx$.

We know that $|sinx| \leq |x|, x \geq 0$ $i.e. -x \leq sinx \leq x$.

$\therefore$ $\lim\limits_{x \to 0}(-x) = 0$ and $\lim\limits_{x \to 0} x = 0.$

$\Rightarrow$ $\lim\limits_{x \to 0} sinx = 0.$

(ii) We know that $-1 \leq cos\left(\frac{1}{x}\right) \leq 1, \forall\, x \neq 0.$

$\therefore$ $-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2, \forall\, x \geq 0.$

$\therefore$ $\lim\limits_{x \to 0}(-x^2) = 0$ and $\lim\limits_{x \to 0}(x^2) = 0.$

$\Rightarrow$ $\lim\limits_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0.$

Example: Find $\lim\limits_{x \to 0} f(x)$, if it exists for the following functions.

$(\boldsymbol{i})$ $f: \mathbb{R} \to \mathbb{R}$ $defined\ by$ $f(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ 1, & x \leq 0 \end{cases}$

Solution: We have $f(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ 1, & x \leq 0 \end{cases}$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 1 = 1.$

$and \ \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(\frac{1}{x}\right)$ does not exists.

$\therefore \lim_{x \to 0} f(x)$ does not exists.

(ii) $f: \mathbb{R} \to \mathbb{R} \ defined \ by \ f(x) = \begin{cases} \frac{1}{x}, & x < 0 \\ 1, & x \geq 0 \end{cases}$

Solution: We have $f(x) = \begin{cases} \frac{1}{x}, & x < 0 \\ 1, & x \geq 0 \end{cases}$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1. \ and \ \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(\frac{1}{x}\right)$ does not exists.

$\therefore \lim_{x \to 0} f(x)$ does not exists.

(iii) $f: [-1, 1] \to \mathbb{R} \ defined \ by \ f(x) = \begin{cases} 0, & -1 \leq x \leq 0 \\ 1, & 0 < x \leq 1 \end{cases}$

Solution: We have $f(x) = \begin{cases} 0, & -1 \leq x \leq 0 \\ 1, & 0 < x \leq 1 \end{cases}$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 0 = 0. \ and$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1$

$\therefore \lim_{x \to 0^-} f(x) \ and \ \lim_{x \to 0^+} f(x)$ both exists.

But they are not equal /same. Therefore, $\lim_{x \to 0} f(x)$ does not exist.

Evaluate (i) $\lim_{x \to \infty} \frac{\sqrt{x-5}}{\sqrt{x+3}}, x > 0.$

Solution: $\lim_{x \to \infty} \frac{\sqrt{x-5}}{\sqrt{x+3}} = \lim_{x \to \infty} \frac{\sqrt{1-\frac{5}{x}}}{\sqrt{1+\frac{3}{x}}}$

$$= \lim_{x \to \infty} \left[\frac{\left(1-\sqrt{\frac{5}{x}}\right)}{\left(1+\sqrt{\frac{3}{x}}\right)}\right] = \frac{\lim_{x \to \infty}\left(1-\sqrt{\frac{5}{x}}\right)}{\lim_{x \to \infty}\left(1+\sqrt{\frac{3}{x}}\right)} = \frac{1-0}{1+0} = 1.$$

(ii) Evaluate $\lim_{x\to\infty} \frac{\sqrt{x}-x}{\sqrt{x}+x}$, $x > 0$.

Solution: $\lim_{x\to\infty} \frac{\sqrt{x}-x}{\sqrt{x}+x} = \lim_{x\to\infty} \left[\frac{\left(\frac{\sqrt{x}}{x}-1\right)}{\left(\frac{\sqrt{x}}{x}+1\right)}\right]$

$$= \lim_{x\to\infty} \left[\frac{\frac{1}{\sqrt{x}}-1}{\frac{1}{\sqrt{x}}+1}\right] = \frac{\lim_{x\to\infty}\left(\frac{1}{\sqrt{x}}-1\right)}{\lim_{x\to\infty}\left(\frac{1}{\sqrt{x}}+1\right)} = \frac{0-1}{0+1} = -1.$$

(iii) Evaluate $\lim_{x\to\infty} \frac{5x^2+3x+20}{3x^2-2x}$.

Solution: $\lim_{x\to\infty} \frac{5x^2+3x+20}{3x^2-2x} = \lim_{x\to\infty} \left[\frac{\left(5+\frac{3}{x}+\frac{20}{x^2}\right)}{\left(3-\frac{2}{x}\right)}\right]$

$$= \frac{\lim_{x\to\infty}\left(5+\frac{3}{x}+\frac{20}{x^2}\right)}{\lim_{x\to\infty}\left(3-\frac{2}{x}\right)} = \frac{5+0+0}{3-0} = \frac{5}{3}.$$

Continuity

Definition:

Let $A \subseteq \mathbb{R}, f: A \to \mathbb{R}, c \in A$. we say that f is continuous at $x = c$ if given $\varepsilon > 0 \, \exists \, \delta > 0$ *such that*

$$\forall \, x \in A, |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon.$$

If function f is not continuous at $x = c$ then it is said to be discontinuous at that point.

Remark:

1. If $c \in A$ is a cluster point of A then a function is continuous at $x = c$ if and only if $\lim_{x\to c} f(x) = f(c)$.

2. If $c \in A$ is not a cluster point of A then f is automatically continuous at c. Such points are often called isolated points of A.

Generally, we test the function for continuity only at cluster points.

Definition : A function f is said to be continuous at a point $x = c$ of its domain if

(i) $\lim_{x\to c} f(x)$ exists. (ii) $f(c)$ is defined. (iii) $\lim_{x\to c} f(x) = f(c)$.

Definition: Let $A \subseteq \mathbb{R}$ *and* $f: A \to \mathbb{R}$. If B is a subset of A, we say that f is continuous on the set B if f is continuous at every point of B.

Example: The constant function $f(x) = b$ is continuous on $\mathbb{R}$.

For, if $c \in \mathbb{R}$ *then* $\lim_{x \to c} f(x) = b = f(c)$. Thus, f is continuous at every point $c \in \mathbb{R}$.

Discontinuous function: A function f which is not continuous is called discontinuous function.

Examples:

1. Let $f(x) = \frac{1}{x}, x \neq 0$.

Here, note that the function is not defined at $x = 0$.

i. e. $f(0)$ is not defined (condition (ii) is not satisfied).

Therefore, function is not continuous at $x = 0$.

2. $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Here, $f(0) = 0$. i. e. function is defined at $x = 0$.

Consider, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{-x}{x} = -1$ *and*

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x}{x} = 1$.

L.H.L.$\neq$ R.H.L.$\Rightarrow \lim_{x \to 0} f(x)$ does not exist.

Therefore, function is not continuous at $x = 0$. (Condition (i) is not satisfied).

3. Let $f(x) = \begin{cases} \frac{sinx}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Here, $f(0) = 0$. i. e. $f(0)$ is defined at $x = 0$. We know that $\lim_{x \to 0} \left(\frac{sinx}{x}\right) = 1 \neq f(0)$.

Therefore, function is not continuous at $x = 0$. (Condition (iii) is not satisfied).

Types of Discontinuity:

There are two types of discontinuities (i) Removable discontinuity (ii) Irremovable OR Essential discontinuity.

Definition: If $\lim_{x \to c} f(x)$ exist but it is not equal to $f(c)$ then we say that $f(x)$ has removable discontinuity at $x = c$.

If $\lim_{x \to c} f(x)$ does not exists then $f(x)$ is said to have Essential/ Irremovable discontinuity.

Examples:

1. Test the continuity of $f(x) = \begin{cases} \frac{sin2x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution: Consider, $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{sin2x}{x} = \lim_{x \to 0} \left(\frac{sin2x}{2x} \times 2\right)$

$$= \lim_{x \to 0} \frac{sin2x}{2x} \times \lim_{x \to 0} 2 = 1.2 = 2.$$

But $f(0) = 0. \therefore \lim_{x \to 0} f(x) \neq f(0)$.

Therefore, f is discontinuous at $x = 0$.

This discontinuity is removable because by redefining the function we can make it as continuous.

i. e. if $f(x) = \begin{cases} \frac{sin2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$ then it is continuous at x = 0.

2. Suppose $f(x) = \begin{cases} \frac{(x^2-16)}{(x-4)}, & x \neq 4 \\ 4, & x = 4 \end{cases}$ then function has removable discontinuity at $x = 4$.

3. Suppose $f(x) = \begin{cases} \frac{x^2-1}{x+1}, & x \neq -1 \\ 2, & x = -1 \end{cases}$ Then function has removable discontinuity at $x = -1$.

4. $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ then this has essential discontinuity.

For, $\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1$ *and* $\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$.

Here, $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x) \ \therefore \ \lim_{x \to 0} f(x)$ does not exist.

5. Discuss the continuity of the function

$(i)\ f(x) = \begin{cases} \frac{x-|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ $\qquad (ii)\ g(x) = \begin{cases} \frac{x-|x|}{x}, & x < 0 \\ 2, & x = 0 \end{cases}$

6. Discuss the continuity of $f(x) = \begin{cases} \frac{x^2}{4} - 4, & 0 < x < 4 \\ 0, & x = 4 \\ 4 - \frac{64}{x^2}, & x > 4 \end{cases}$

7. Check the continuity of $f(x) = \begin{cases} \frac{\left(e^{\frac{1}{x}} - e^{-\frac{1}{x}}\right)}{\left(e^{\frac{1}{x}} + e^{-\frac{1}{x}}\right)}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

8. Check the continuity of $h(x) = \begin{cases} \frac{e^{\frac{1}{x}} + 1}{e^{\frac{1}{x}} - 1}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

Continuity at end points:

Definition: Let a function f be defined on a closed interval [a, b]. Then f is said to be continuous at $x = a$ if it is continuous from right at $x = a$. i. e. if $\lim_{x \to a^+} f(x) = f(a)$ and f is said to be continuous at $x = b$ if it is continuous from left at $x = b$.

$i.e.$ if $\lim_{x \to b^-} f(x) = f(b)$.

Continuity of a function on an interval:

Definition: Let a function f be defined on a closed interval [a, b]. Then f is said to be continuous on the closed interval [a, b], if

(i) $\lim_{x \to c} f(x) = f(c)$, $\forall\, c \in (a, b)$ i. e. f is continuous at every point of the interval (a, b).

(ii) $\lim_{x \to a^+} f(x) = f(a)$ and

(iii) $\lim_{x \to b^-} f(x) = f(b)$

Algebra of continuous functions:

Theorem 7: Let $A \subseteq \mathbb{R}$, let f and g are continuous functions at $x = c$ then

(a) $f \pm g, kf, f.g$ are continuous at $x = c$, k- constant.

(b) $g: A \to \mathbb{R}$ is continuous at $c \in A$ and if $g(x) \neq 0, for\ all\ x \in A$ then the quotient function $\frac{f}{g}$ is continuous at $x = c$.

Proof: Since f and g are continuous at $x = c$.

$\therefore \lim_{x\to c} f(x) = f(c)$ and $\lim_{x\to c} g(x) = g(c)$.

(a) Consider, $\lim_{x\to c}(f \pm g)(x) = \lim_{x\to c}\big(f(x) \pm g(x)\big)$

$$= \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x) = f(c) \pm g(c)$$

$$= (f \pm g)(c).$$

Therefore, $f \pm g$ is continuous at $x = c$.

Consider, $\lim_{x\to c}(kf)(x) = \lim_{x\to c}\big(k. f(x)\big) = k \lim_{x\to c} f(x)$

$$= k. f(c) = (kf)(c).$$

i. e. kf is continuous at $x = c$.

Consider, $\lim_{x\to c}(f. g)(x) = \lim_{x\to c} f(x). g(x) = \lim_{x\to c} f(x) . \lim_{x\to c} g(x)$

$$= f(c). g(c) = (fg)(c).$$

Therefore, $f. g$ is continuous at $x = c$.

(b) Since $c \in A$ $g(c) \neq 0$. But as $\lim_{x\to c} g(x) = g(c)$.

We have $\lim_{x\to c}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{f(c)}{g(c)} = \left(\frac{f}{g}\right)(c)$.

Therefore, $\frac{f}{g}$ is continuous at $x = c$.

Theorem 8: If f is continuous function at $x = c$ then $|f|$ is also continuous at $x = c$.

Proof: We shall use the $\varepsilon - \delta$ definition of continuity to prove this theorem. Suppose f is continuous function at $x = c$. then by the definition, for given $\varepsilon > 0 \ \exists \ \delta > 0 \ s.t.$

$|f(x) - f(c)| < \varepsilon$ $whenever$ $|x - c| < \delta$.

Consider, $\big||f(x)| - |f(c)|\big| \leq |f(x) - f(c)| < \varepsilon,$ $whenever$ $|x - c| < \delta$.

Therefore, by the definition, $|f(x)|$ is continuous at $x = c$.

Remark: The converse of this theorem is not true.

For, Let $f(x) = \begin{cases} -1, & x < c \\ 1, & x \geq c \end{cases}$

Then $\lim_{x \to c} |f(x)| = \lim_{x \to c} (1) = 1 = f(c)$. Therefore, $|f(x)|$ is continuous at $x = c$.

But $\lim_{x \to c} f(x)$ does not exist.

Because $\lim_{x \to c^-} f(x) = -1 \; and \; \lim_{x \to c^+} f(x) = 1$.

Therefore, f is not continuous at $x = c$.

Theorem 9: If f is continuous at $x = c \; and \; f(c) \geq 0 \; then \; \sqrt{f}$ is continuous at $x = c$.

Proof: As f is continuous at $x = c$,

$\lim_{x \to c} f(x) = f(c). \lim_{x \to c} \left(\sqrt{f(x)}\right) = \sqrt{\lim_{x \to c} f(x)} = \sqrt{f(c)}$.

Therefore, $\sqrt{f}$ is continuous at $x = c$.

Composition of Continuous functions:

Theorem 10: Let $A \subseteq \mathbb{R}, f: A \to \mathbb{R} \; and \; g: B \to \mathbb{R}$ be functions such that $f(A) \subseteq B$. If f is continuous at a point $c \in A$ and g is continuous at $b = f(c) \in B$ then the composite function $g \circ f: A \to \mathbb{R}$ is continuous at c.

Proof: Since f is continuous at $c \in A$.

$\therefore \; for \; \varepsilon > 0 \; \exists \; \delta > 0 \; s.t. |x - c| < \delta \Rightarrow \; |f(x) - f(c)| < \rho$ .... (1)

Also, g is continuous at $b = f(c) \in B$.

$\therefore \; for \; \varepsilon > 0, \exists \; \rho > 0 \; s.t. |f(x) - f(c)| < \rho$

$\Rightarrow \; |g(f(x)) - g(f(c))| < \varepsilon$.......... (2)

From Eqn. (1) and (2), we have for given $\varepsilon > 0, \exists \; \delta > 0 \; s.t.$

$$when \; |x - c| < \delta \Rightarrow \; |g(f(x)) - g(f(c))| < \varepsilon$$

$\Rightarrow |x - c| < \delta \Rightarrow \; |(g \circ f)(x) - (g \circ f)(c)| < \varepsilon$.

This shows that the composite function $g \circ f$ is continuous at $x = c$.

Examples:

1. Find $\alpha \; and \; \beta$, if the function $f(x)$ is continuous on $(-3, 5)$;

$$\text{where } f(x) = \begin{cases} x + \alpha, & -3 < x < 1 \\ 3x + 2, & 1 \leq x < 3 \\ \beta + x, & 3 \leq x < 5 \end{cases}$$

Solution: We shall test the continuity of $f(x)\ at\ x = 1\ and\ 3$.

(a) At $x = 1$:

$f(1) = 5, when\ x = 1\ f(x) = 3x + 2.$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(3x + 2) = 3(1) + 2 = 5.$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x + \alpha) = 1 + \alpha.$

But it is given that $f(x)$ is continuous at $x = 1$.

$\therefore R.H.L. = L.H.L. \therefore 5 = 1 + \alpha \Rightarrow \alpha = 4.$

(b) At $x = 3$:

$f(3) = \beta + 3, when\ x = 3\ f(x) = \beta + 3.$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(\beta + x) = \beta + 3.$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(3x + 2) = 3(3) + 2 = 11.$

But it is given that $f(x)$ is continuous at $x = 3$.

$\therefore R.H.L. = L.H.L. \therefore\ \beta + 3 = 11 \Rightarrow \beta = 8.$

2. Find α, β if the function is continuous on (-2, 2); where $f(x) = \begin{cases} x + \alpha, & -2 < x < 0 \\ 2x + 1, & 0 \le x < 1 \\ \beta - x, & 1 \le x < 2 \end{cases}$

3. Discuss the continuity of $f(x)$ at $x = 1, 2, 4$; where

$$f(x) = \begin{cases} 2x - 1, & x \le 1 \\ x^2, & 1 < x < 2 \\ 3x - 4, & 2 \le x < 4 \\ x^{\frac{3}{2}}, & x \ge 4 \end{cases}$$

Solution: At the point $x = 1$:

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x^2) = 1\ and$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(2x - 1) = 2(1) - 1 = 1.$

$Also\ we\ when\ x = 1, f(x) = 2x - 1\ \therefore f(1) = 2(1) - 1 = 1.$

$\therefore we\ have\ \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1).$

$\therefore f(x)$is continuous at $x = 1$.

At point $x = 2$:

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3x-4) = 3(2)-4 = 2.$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2) = 2^2 = 4.$

$\therefore \lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$

$\therefore \lim_{x\to 2} f(x)$ does not exist.

$\therefore f(x)$ is not continuous at $x = 2$.

At the point $x = 3$:

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} (x^{\frac{3}{2}}) = \left(4^{\frac{3}{2}}\right) = (2^3) = 8.$

$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (3x-4) = 3(4)-4 = 8.$

$Also\ when\ x = 4,\ f(x) = \left(x^{\frac{3}{2}}\right) \therefore f(4) = 4^{\frac{3}{2}} = 8.$

$\therefore we\ have\ \lim_{x\to 4^+} f(x) = \lim_{x\to 4^-} f(x) = f(4).$

$\therefore f(x)$ is continuous at $x = 4$.

4. The function f is defined on [0, 3] by $f(x) = \begin{cases} x^2, & 0 \le x < 1 \\ 1+x, & 1 \le x \le 2 \\ \frac{6}{x}, & 2 < x \le 3 \end{cases}$ Discuss the continuity of $f(x)\ on\ [0,3]$.

5. Discuss the continuity of function $f(x) = \begin{cases} x^2+2, & 0 \le x < 1 \\ 4x-1, & 1 \le x \le 2 \\ x^2-1, & 2 < x \le 3 \end{cases}$

6. Test the function for continuity on [-2, 2]; where

$$f(x) = \begin{cases} 2-3x, & -2 \le x \le 1 \\ 2x+7, & -1 < x < 1 \\ 4x+1, & 1 \le x \le 2 \end{cases}$$

Solution: We need to check the continuity of $f(x)\ at$

$x = -1, 1\ and\ \lim_{x\to 2^-} f(x), \lim_{x\to -2^+} f(x).$

At the point $x = -1$:

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (2x+7) = 2(-1)+7 = 5.$

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (2-3x) = 2-3(-1) = 5.$

$Also\ when\ x = -1, f(x) = 2 - 3x\ \therefore f(-1) = 2 - 3(-1) = 5.$

$\therefore \lim_{x\to -1^+} f(x) = \lim_{x\to -1^-} f(x) = f(-1).$

$\therefore f(x)$ is continuous at $x = -1$.

At the point $x = 1$:

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (4x + 1) = 4(1) + 1 = 5.$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (2x + 7) = 2(1) + 7 = 9.$

$\therefore \lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x).$

$\therefore \lim_{x\to 1} f(x)$ does not exist.

$\therefore f(x)$ is not continuous at $x = 1$.

Consider,

$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} (2 - 3x) = 2 - 3(-2) = 8.$

$When\ x = -2, f(x) = 2 - 3x\ \therefore f(-2) = 2 - 3(-2) = 8.$

$\therefore \lim_{x\to -2^+} f(x) = f(-2).$ $\therefore f(x)$ is continuous from right at $x = -2$.

Again consider,

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (4x + 1) = 4(2) + 1 = 9.$

$When\ x = 2, f(x) = 4x + 1. \therefore f(2) = 4(2) + 1 = 9.\ \therefore \lim_{x\to 2^-} f(x) = f(2).$

$\therefore f(x)$ is continuous from left at $x = 2$.

Thus, the given function is continuous everywhere on [-2, 2] except at $x = 1$.

7. Find a, b so that the given function will be continuous for every x.

(i) $f(x) = \begin{cases} ax + 3, & x > 5 \\ 8, & x = 5 \\ x^2 + bx + 1, & x < 5 \end{cases}$

(ii) $f(x) = \begin{cases} \sqrt{3}, & x = 0 \\ 2\sin(\cos^{-1} x), & 0 < x < 1 \\ ax + b, & x < 0 \end{cases}$

(iii) $g(x) = \begin{cases} \frac{\sqrt{x} - a}{x - 1}, & x > 1 \\ b, & x \leq 1 \end{cases}$

(iv) If $f(x)$ is continuous at $x = 0$ and $f(1) = 2, find\ a\ and\ b$; where

$$f(x) = \begin{cases} x^2 + a, & x \geq 0 \\ 2\sqrt{x^2+2} + b, & x \leq 0 \end{cases}$$

Solution: We have $f(x) = \begin{cases} x^2 + a, & x \geq 0 \\ 2\sqrt{x^2+2} + b, & x \leq 0 \end{cases}$ $and\ f(1) = 2.$

$\therefore (1^2) + a = 2 \Rightarrow a = 1.$

Now, $f(x)$ is continuous at $x = 0$.

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0). \therefore \lim_{x\to 0^+}(x^2 + a) = \lim_{x\to 0^-}\left(2\sqrt{x^2+2} + b\right)$

$\Rightarrow a = 2\sqrt{2} + b \Rightarrow b = 1 - 2\sqrt{2}.$

8. Let f be defined for all $x \in \mathbb{R}, x \neq 2, by\ f(x) = \frac{x^2+x-6}{x-2}$.

Can f be defined so that function is continuous at that point?

Solution: We have, a function is defined at point $x = 2$.

Now, $f(x) = \frac{x^2+x-6}{x-2} = \frac{(x-2)(x+3)}{x-2} = (x + 3).$

Therefore, if we define $f(2) = 5$ $then$ $\lim_{x\to 2} f(x) = \lim_{x\to 2}(x + 3) = 5 = f(2).$

So that the function is continuous at $x = 2$.

9. Examine the continuity of $f(x) = \sqrt{\frac{x-1}{x+3}}$.

Solution: Let $\emptyset(x) = \frac{x-1}{x+3}$ $and\ \varphi(y) = \sqrt{y}$ then $f(x) = \varphi\big(\emptyset(x)\big)$. By Composite function theorem, if $\emptyset(x)$ is continuous at x and $\varphi(y)$ is continuous at $\emptyset(x)$ then $\varphi_0\emptyset$ is continuous at x.

Consider $\emptyset(x) = \frac{x-1}{x+3}$. We observe that for all values of x except $x = -3$ it is continuous.

Further, $\varphi(y)$ is continuous $\forall\ y = \emptyset(x) \geq 0.\ i.e. \frac{x-1}{x+3} \geq 0$. Which is possible only when

(i) $(x - 1) \geq 0\ and\ (x + 3) \geq 0\ \ i.e.\ if\ x \geq 1\ and\ x > -3\ \therefore x \geq 1$ and

(ii) $(x - 1) \leq 0\ and\ (x + 3) < 0\ \ i.e.\ if\ x \leq 1\ and\ x < -3\ \therefore x < -3.$

Hence, $f(x)$ is continuous on $(-\infty, -3)$ and $[1, \infty)$.

Definition: A function $f: A \to \mathbb{R}$ is said to be bounded on A if there exists a constant $M > 0$ $such\ that\ |f(x)| \leq M\ for\ all\ x \in A$.

Theorem 11: (Boundedness Theorem)-

Let $I = [a, b]$ be a closed and bounded interval and $f: I \to \mathbb{R}$ be continuous function on I. Then f is bounded on I.

Definition: Let $A \subseteq \mathbb{R}\ and\ let\ f: A \to \mathbb{R}$ be a function defined on A. We say that f has an absolute maximum on A if there exist a point $x_1 \in A\ such\ that\ f(x_1) \geq f(x), \forall\ x \in A$. And we say that f has an absolute minima on A if there exist a point $x_1 \in A\ such\ that$

$$f(x_1) \leq f(x), \forall\ x \in A.$$

Theorem 12: (Maximum – Minimum Theorem)-

Let $I = [a, b]$ be a closed bounded interval and let $f: I \to \mathbb{R}$ be a continuous function on I. Then f has an absolute maximum and an absolute minimum on I.

Theorem 13: (Location of Roots theorem)-

Let $I = [a, b]$ be a closed bounded interval and let $f: I \to \mathbb{R}$ be a continuous function on I. If $f(a) < 0 < f(b)\ \ OR\ \ f(a) > 0 > f(b)$ then there exists a number $c \in (a, b)\ s.t. f(c) = 0$.

Theorem 14: (Bolzano's Intermediate Value Theorem)-

Let $I = [a, b]$ be a closed bounded interval and let $f: I \to \mathbb{R}$ be a continuous function on I. If $a, b \in I\ and\ k \in \mathbb{R}$ such that $f(a) < k < f(b)$ then there exists a point $c \in I$ between $a\ and\ b\ such\ that\ f(c) = k$.

Proof: Suppose that $a < b\ and\ let\ g(x) = f(x) - k\ then\ g(a) < 0 < g(b)$. By the location of roots theorem, there exists a point $c \in I\ with\ \ a < c < b\ such\ that$

$$g(c) = f(c) - k = 0.\ \therefore f(c) = k.$$

Now, suppose that $b < a\ and\ let\ h(x) = k - f(x)\ then\ h(b) < 0 < h(a)$. By the location of roots theorem, there exists a point $c \in I\ with\ b < c < a\ such\ that$

$$h(c) = k - f(c) = 0.\ \therefore f(c) = k.$$

Theorem 15: Let $I = [a, b]$ be a closed bounded interval and let $f: I \to \mathbb{R}$ be a continuous function on I. If $k \in \mathbb{R}$ any number satisfying $Inf\ f(I) \leq k \leq Sup\ f(I)$ then there exists a point $c \in I$ between $a\ and\ b\ such\ that\ f(c) = k$.

Proof: By Maximum-Minimum theorem, there are points $c_1 and\ c_2\ in\ I\ such\ that$

$Inf\ f(I) = f(c_2) \leq k \leq f(c_1) = Sup\ f(I)$. Hence, there exists a point $c \in I\ s.t.$

$f(c) = k$.

Example:

1. Give an example of functions $f\ and\ g$ that are both discontinuous at a point $c \in \mathbb{R}\ s.t.$

(a) The sum $f + g$ is continuous at c.

(b) The product $f.g$ is continuous at c.

Solution: Let us define the functions $f(x) and\ g(x)$ as-

$f(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$ $and\ g(x) = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0 \end{cases}$. Then both the functions are discontinuous at

x = 0.

(a) We have $(f + g)(x) = f(x) + g(x) = 1, \forall\, x \in \mathbb{R}$, which is a constant function and hence it is continuous for all $x \in \mathbb{R}$.

(b) We have$(f.g)(x) = f(x).g(x) = 0, \forall\, x \in \mathbb{R}$, which is a constant function and hence it is continuous for all $x \in \mathbb{R}$.

2. Let $I = [a, b]$ and let $f: I \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for each x in I. Prove that there exists a number $\alpha > 0\ s.t. f(x) \geq \alpha, for\ all\ x \in I.$

Solution: Since f is continuous on closed and bounded interval I = [a, b]. By Max. – Min. theorem , there exists $x_1 \in I\ s.t. f(x_1) \leq f(x), \forall\, x \in I.$

Now, $f(x) > 0, \forall\, x \in I\ \Rightarrow f(x_1) > 0$. If we set $f(x_1) = \alpha$ then $f(x) > \alpha$.

Exercise

1. By using the definition of limit of a function prove the following.

$$(i)\ \lim_{x\to 1}(2x + 4) = 9 \quad (ii)\ \lim_{x\to 2}\left(\frac{x^2 - 3x + 2}{x - 2}\right) = 1$$

$$(iii)\ \lim_{x\to 1}(x^3) = 1 \quad (iv)\ \lim_{x\to 0}\left(\frac{5x + 7}{3x + 1}\right) = 7.$$

2. Show that $\lim_{x\to 1} f(x)$ does not exist. Where $f(x) = \begin{cases} x + 1, & 0 \leq x < 1 \\ 2, & x = 1 \\ 2 - x, & 1 < x \leq 2 \end{cases}$
3. If $f(x) = \frac{|x|}{x}$, $x \neq 0$, show that $\lim_{x\to 0} f(x)$ does not exist.
4. Discuss the continuity of $f(x) = \begin{cases} \frac{x^2-9}{x-3}, & 0 \leq x < 3 \\ 4x - 6, & 3 < x \leq 6 \end{cases}$
5. Test the continuity of $f(x) = \begin{cases} x, & 0 \leq x \leq \frac{1}{2} \\ 1, & x = \frac{1}{2} \\ 1 - x, & \frac{1}{2} < x < 1 \end{cases}$

6. Examine for the continuity $(i)\ f(x) = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right), & x \neq 0 \\ \frac{\pi}{4}, & x = 0 \end{cases}$

$$(ii)\ f(x) = \begin{cases} 1 + x, & -1 \leq x < 0 \\ 1, & x = 0 \\ 1 - x, & 0 < x \leq 1 \end{cases}$$

7. Draw the graphs of the following functions;

$(i)\ f(x) = \frac{1}{4}x^2 - x$ $(ii)\ f(x) = \frac{1}{9}x^2 - x$ $(iii)\ g(x) = 3x^2 - 7$

$(iv)\ G(x) = \sqrt{4 - x^2}$ $(v)\ f(x) = \frac{x^2-4}{x-2}$ $(vi)\ h(x) = \frac{4}{x}$

$(vii)\ F(x) = x - |x|$ $(viii)\ f(x) = 4x^2$ $(ix)\ f(x) = 4x^2 + 2$

$(x)\ G(x) = |x - 2|$ $(xi)\ h(x) = |x| + 2$ (xii) Consider the function $f: [-3, 3] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 0, & -3 \leq x < -2 \\ 1, & -2 \leq x < -1 \\ 2, & -1 < x < 1 \\ 3, & 1 \leq x < 2 \\ -2, & 2 \leq x < 3 \end{cases}$ Draw the graph of $f(x)$.

Answers: 2. Continuous at $= \frac{1}{2}$.

3. Discontinuous at $x = \frac{1}{2}$.

4. (i) Discontinuous at $x = 0$. (ii) Continuous at $x = 0$.

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