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Limit

Sub : Multivariable Calculus I

Class : S. Y. B. Sc.

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Limit of a function of two variables:

A function $f(x,y)$ has a limit L as (x,y) approaches (x_0,y_0), if for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$|f(x,y) - L| < \varepsilon$ whenever $0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$

Or $0 < |x - x_0| < \delta$ *and* $0 < |y - y_0| < \delta$

We can write it as

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

This limit is called as simultaneous limit.

Repeated limit or Iterated limit:

If $f(x,y)$ is defined in a certain deleted-neighbourhood of (x_0,y_0) and if $\lim_{x\to x_0} f(x,y)$ exists, then it is a function of y say $\phi(y)$ also if $\lim_{y\to y_0} \phi(y)$ exists and equal to L_1. Then L_1 is called as repeated limit is written as

$$\lim_{y \to y_0} \left\{ \lim_{x \to x_0} f(x, y) \right\} = \lim_{y \to y_0} \phi(y) = L_1$$

Or

$$\lim_{y \to y_0} \lim_{x \to x_0} f(x, y) = L_1$$

Similarly other repeated limit can be written as

$$\lim_{x \to x_0} \left\{ \lim_{y \to y_0} f(x, y) \right\} = \lim_{x \to x_0} \psi(y) = L_2$$

Or

$$\lim_{x \to x_0} \lim_{y \to y_0} f(x, y) = L_2.$$

Example1: By using $\varepsilon - \delta$ definition show that

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+1} = 0\,.$$

Solution: Let $f(x,y) = \frac{x+y}{x^2+1}$

since $\qquad x^2 \geq 0 \Rightarrow x^2 + 1 \geq 1 \Rightarrow \frac{1}{x^2+1} \leq 1$

Also, $\qquad x^2 \leq x^2 + y^2 \Rightarrow |x| \leq \sqrt{x^2+y^2}$

And $\qquad |y| \leq \sqrt{x^2+y^2}$

Let $\varepsilon > 0$ be given

If $\quad 0 < \sqrt{x^2+y^2} < \delta$ then,

Consider

$$|f(x,y) - 0| = \left|\frac{x+y}{x^2+1} - 0\right| = \left|\frac{x+y}{x^2+1}\right| \leq \frac{|x|+|y|}{x^2+1} \leq |x| + |y|$$

$$\leq \sqrt{x^2+y^2} + \sqrt{x^2+y^2} < \delta + \delta = 2\delta = \varepsilon$$

$\therefore$ for given $\varepsilon > 0$, $\exists \delta = \frac{\varepsilon}{2}$ such that

$|f(x,y) - L| < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+1} = 0\,.$$

Example2: By using $\varepsilon - \delta$ definition show that

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0 .$$

Solution: Let $f(x,y) = \frac{4xy^2}{x^2+y^2}$

Since, $|x| \le \sqrt{x^2+y^2}$ and $y^2 \le x^2+y^2 \Rightarrow \frac{y^2}{x^2+y^2} \le 1$

Let $\varepsilon > 0$ be given

If $\quad 0 < \sqrt{x^2+y^2} < \delta$ then,

Consider

$$|f(x,y) - 0| = \left|\frac{4xy^2}{x^2+y^2} - 0\right| = \left|\frac{4xy^2}{x^2+y^2}\right| \le 4|x| \le \sqrt{x^2+y^2} < \delta = \varepsilon$$

$\therefore$ for given $\varepsilon > 0$, $\exists \delta = \varepsilon$ such that

$|f(x,y) - L| < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$

$$\therefore \quad \lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0 .$$

Limit Along Path

Example3:Examine whether the limit exist. If exist find it.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^3}$$

Solution: Taking limit along $y = mx$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^3} = \lim_{x\to 0} \frac{x^2(mx)}{x^3+(mx)^3}$$

$$= \lim_{x\to 0} \frac{mx^3}{x^3+m^3x^3}$$

$$= \lim_{x\to 0} \frac{m}{1+m^3}$$

$$= \frac{m}{1+m^3}$$

For different values of m we get different limit therefore limit does not exist

Example4:Example3:Examine whether the limit exist. If exist find it.

$$\lim_{(x,y)\to(1,2)} \frac{sin^{-1}(xy-2)}{tan^{-1}(5xy-10)}$$

Solution: $\lim_{(x,y)\to(1,2)} \frac{sin^{-1}(xy-2)}{tan^{-1}(5xy-10)} = \lim_{(x,y)\to(1,2)} \frac{sin^{-1}(xy-2)}{tan^{-1}(5xy-10)} \times \frac{(xy-2)}{(xy-2)}$

$$= \lim_{(x,y)\to(1,2)} \frac{sin^{-1}(xy-2)}{(xy-2)} \times \frac{(xy-2)}{tan^{-1}5(xy-2)}$$

$$= 1 \times \frac{1}{5} = \frac{1}{5}.$$

Example5:Test the existence of simultaneous limit and repeated limits of the following function at origin.

$$f(x,y) = \frac{x-y}{x+y}$$

Solution: Repeated limit

$$\lim_{x\to0} \lim_{y\to0} \frac{x-y}{x+y} = \lim_{x\to0} \frac{x-0}{x+0} = \lim_{x\to0} 1 = 1$$

$$\lim_{y\to 0}\lim_{x\to 0}\frac{x-y}{x+y}=\lim_{y\to 0}\frac{0-y}{0+y}=\lim_{y\to 0}(-1)=-1$$

Both repeated limit exists but not equal

Simultaneous limit

$$\lim_{(x,y)\to(0,0)}\frac{x-y}{x+y}$$

Taking limit along $y=mx$

$$\therefore \lim_{(x,y)\to(0,0)}\frac{x-y}{x+y}=\lim_{x\to 0}\frac{x-mx}{x+mx}$$

$$=\lim_{x\to 0}\frac{1-m}{1+m}$$

$$=\frac{1-m}{1+m}$$

Limit depends on m therefore simultaneous limit does not exist.

Example6: Show that $\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$ does not exist by considering different paths.

Solution: 1) take limit along X$-axis$ i.e. $y = 0$

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2} = \lim_{y\to 0} \frac{x^4}{x^4+y^2} = \frac{x^4}{x^4} = 1$$

2) take limit along $Y - axis$ i.e. $x = 0$

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2} = \lim_{x\to 0} \frac{x^4}{x^4+y^2} = 0$$

For two different paths we get two different limits, therefore limit does not exist.

Example6: Show that $\lim_{(x,y)\to(0,0)} \frac{x^2+y}{y}$ does not exist by considering different paths.

Solution: 1) taking limit along $Y - axis$ i.e. $x = 0$

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y}{y} = \lim_{x\to 0} \frac{x^2+y}{y} = \frac{0+y}{y} = 1$$

2) Taking limit along $y = x^2$

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y}{y} = \lim_{y\to x^2} \frac{x^2+y}{y} = \frac{x^2+x^2}{x^2} = 2$$

For two different paths we get two different limits, therefore limit does not exist.

Substitution of Polar Co-ordinates

If it is difficult to find the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ in rectangular co-ordinates then use the substitution $x = rcos\,\theta\ and\ y = rsin\,\theta$. In that case $(x,y) \to (0,0)$ is equivalent to $r \to 0$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r\cos\theta, r\sin\theta)$$

Theorem: If a function f is bounded in a deleted neighbourhood of $(x_0, y_0)$and $\lim_{(x,y)\to(0,0)} g(x,y) = 0$, then

$$\lim_{(x,y)\to(0,0)} f(x,y)\cdot g(x,y) = 0$$

Example7: Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^3-xy^2}{x^2+y^2}$, if exist.

Solution: Put $x = rcos\,\theta\ and\ y = rsin\,\theta$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^3-xy^2}{x^2+y^2} = \lim_{r\to0} \frac{r^3cos^3\theta - rcos\,\theta r^2 sin^2\theta}{r^2(cos^2\theta + sin^2\theta)}$$

$$= \lim_{r\to0} \frac{r^3(cos^3\theta - cos\,\theta sin^2\theta)}{r^2}$$

$$= \lim_{r\to0} r(cos^3\theta - cos\,\theta sin^2\theta)$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^3-xy^2}{x^2+y^2} = 0$$

Since $\lim_{r\to0} r = 0$ and

$|cos^3\theta - cos\,\theta sin^2\theta| \le |cos^3\theta| + |\cos\theta||sin^2\theta| \le 1 + 1 = 2$

i.e. $(cos^3\theta - cos\,\theta sin^2\theta)$ is bounded.

Exercise

1) By using $\varepsilon - \delta$ definition show that

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{2+\cos x}.$$

2) Evaluate the $\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2}$

3) By using different paths show that $\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$ does not exist.

4) Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$.

5) Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^6+y^4}$

6) Evaluate $\lim_{(x,y)\to(2,1)} \frac{sin^{-1}(xy-2)}{tan^{-1}(3xy-6)}$

References : Text book of Multivariable Calculus I prepared by B. O. S. in Mathematics, Savitribai Phule Pune University, Pune.

Thank You