K. T. S. P. Mandal's

Hutatma Rajguru Mahavidyalaya, Rajgurunagar

Tal-Khed, Dist-Pune (410 505)

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Subject – Vector Calculus

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Chapter 2: Integrals

Topic- Line integral of vector field

Prepared by

Prof. R. M. Wayal

Department of Mathematics

Hutatma Rajguru Mahavidyalaya, Rajgurunagar

* **Vector field -**

A vector field on two or three dimensional space is a function $\bar{F}$ that assign to each point (x,y) or (x,y,z) a two dimensional or three dimensional vector given by $\bar{F}(x,y)$ or $\bar{F}(x,y,z)$ resp.

The standard notation for the function $\bar{F}$ is

$$\bar{F}(x,y) = P(x,y)\,\bar{i} + Q(x,y)\,\bar{j}$$

$$\bar{F}(x,y,z) = P(x,y,z)\,\bar{i} + Q(x,y,z)\,\bar{j} + R(x,y,z)\,\bar{k}$$

* **Gradient vector field -** (grad F)

If $f(x,y,z)$ is differentiable scalar valued function then a vector field

$$\nabla f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k}$$

is called a gradient vector field.

* **Line integral of vector field -**

Let $\bar{F}$ be a vector field with continuous components defined along a smooth curve c parametrized by $\bar{r}(t)$, $a \le t \le b$. Then the line integral of $\bar{F}$ along c is defined as

$$\int_c \bar{F}\cdot\bar{T}\,ds = \int_c \left(\bar{F}\cdot\frac{d\bar{r}}{ds}\right)\cdot ds = \int_c \bar{F}\cdot d\bar{r}$$

Line integral - $\int_c \bar{F}\cdot d\bar{r}$

Ex. Find the gradient field of the function

$$f(x,y,z) = xy + yz + xz$$

$$\Rightarrow \text{grad } f = \nabla f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k}$$

$$\nabla f = (y+z)\,\bar{i} + (x+z)\,\bar{j} + (y+x)\,\bar{k}$$

Ex. $f(x,y,z) = e^z - \log(x^2+y^2)$

$$\nabla f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k}$$

$$= -\frac{1}{x^2+y^2}2x\,\bar{i} - \frac{1}{x^2+y^2}2y\,\bar{j} + e^2\,\bar{k}$$

Ex. $f(x,y,z) = (x^2+y^2+z^2)^{-\frac{1}{2}}$

$$\text{grad} f = \nabla f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k}$$

$(\;)^n = n(\;)^{n-1}(\;)'$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}-1}(2x)\,\bar{i}$$

$$-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2y)\,\bar{j}$$

$$-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2z)\,\bar{k}$$

$$= -x(x^2+y^2+z^2)^{-3/2}\,\bar{i} - y(x^2+y^2+z^2)^{-3/2}\,\bar{j}$$

$$-z(x^2+y^2+z^2)^{-3/2}\,\bar{k}$$

Ex. Evaluate $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F}(x,y,z) = z\bar{i} + xy\bar{j} - y^2\bar{k}$

along the curve c given by

$$\bar{r}(t) = t^2\,\bar{i} + t\,\bar{j} + \sqrt{t}\,\bar{k} \quad , \quad 0 \le t \le 1$$

$\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

$\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

$\bar{a}\cdot\bar{b} = a_1b_1 + a_2b_2 + a_3b_3$

$$\Rightarrow \quad \bar{r}(t) = t^2\,\bar{i} + t\,\bar{j} + \sqrt{t}\,\bar{k}$$

$$d\bar{r} = 2t\,dt\,\bar{i} + dt\,\bar{j} + \frac{1}{2\sqrt{t}}dt\,\bar{k}$$

$$x = t^2,\; y = t,\; z = \sqrt{t}$$

$$\therefore \bar{F} = \sqrt{t}\,\bar{i} + t^2\cdot t\,\bar{j} - t^2\bar{k} = \sqrt{t}\,\bar{i} + t^3\bar{j} - t^2\bar{k}$$

$$\bar{F}\cdot d\bar{r} = \sqrt{t}(2t\,dt) + t^3(dt) + (-t^2)\frac{1}{2\sqrt{t}}dt$$

$$= \left[2t^{3/2} + t^3 - \frac{1}{2}t^{3/2}\right]dt$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 \left[2t^{3/2} + t^3 - \frac{1}{2}t^{3/2}\right]dt$$

$$= \left[2\frac{t^{5/2}}{5/2} + \frac{t^4}{4} - \frac{1}{2}\frac{t^{5/2}}{5/2}\right]_0^1$$

$$= \left(\frac{4}{5} + \frac{1}{4} - \frac{1}{5}\right) - 0$$

$$= \frac{3}{5} + \frac{1}{4} = \frac{12+5}{20} = \frac{17}{20}$$

Ex. $\int_C \bar{F} \cdot d\bar{r}$, where $\bar{F}(x,y,z) = e^{2x}\,\bar{i} + z(y+1)\,\bar{j} + z^3\,\bar{k}$

and c is given by $\bar{r}(t) = t^3\,\bar{i} + (1-3t)\,\bar{j} + e^t\,\bar{k}$, $0 \le t \le 2$

$$\Rightarrow \quad \bar{r}(t) = t^3\,\bar{i} + (1-3t)\,\bar{j} + e^t\,\bar{k}$$

$$d\bar{r} = 3t^2\,dt\,\bar{i} - 3\,dt\,\bar{j} + e^t\,dt\,\bar{k}$$

$$x = t^3, \quad y = 1-3t, \quad z = e^t$$

$$\bar{F} = e^{2t^3}\,\bar{i} + e^t(1-3t+1)\,\bar{j} + (e^t)^3\,\bar{k}$$

$$= e^{2t^3}\,\bar{i} + e^t(2-3t)\,\bar{j} + e^{3t}\,\bar{k}$$

$$\bar{F} \cdot d\bar{r} = \left[3t^2 e^{2t^3} - 3e^t(2-3t) + e^t \cdot e^{3t}\right] dt$$

$$\int_C \bar{F} \cdot d\bar{r} = \int_0^2 3t^2 e^{2t^3}\,dt - 6\int_0^2 e^t\,dt + 9\int_0^2 e^t \cdot t\,dt + \int_0^2 e^{4t}\,dt$$

put $t^3 = m$

$\Rightarrow 3t^2 dt = dm$

$t=0 \Rightarrow m=0$ & $t=2 \Rightarrow m=8$

$\int pf = p\int f - p'\int\int f + p''\int\int\int f$

$$= \int_0^8 e^{2m}\,dm - 6\left[e^t\right]_0^2 + 9\left[t e^t - 1 e^t\right]_0^2 + \left[\frac{e^{4t}}{4}\right]_0^2$$

$$= \left[\frac{e^{2m}}{2}\right]_0^8 - 6\left[e^2 - e^0\right] + 9\left[(2e^2 - e^2) - (-e^0)\right] + \frac{e^8}{4} - \frac{e^0}{4}$$

$$= \frac{e^{16}}{2} - \frac{1}{2} - 6e^2 + 6 + 9e^2 + 9 + \frac{e^8}{4} + \frac{1}{4}$$

Ex. $\int \bar{F}\cdot d\bar{r}$, $\bar{F}(x,y,z) = 8x^2yz\,\bar{i} + 5z\,\bar{j} - 4xy\,\bar{k}$ and c is the curve given by $\bar{r}(t) = t\,\bar{i} + t^2\bar{j} + t^3\bar{k}$ $0 \le t \le 1$.

$\Rightarrow$ $\bar{r}(t) = t\,\bar{i} + t^2\,\bar{j} + t^3\,\bar{k}$

$d\bar{r} = dt\,\bar{i} + 2t\,dt\,\bar{j} + 3t^2dt\,\bar{k}$

$x = t,\ y = t^2,\ z = t^3$

$\bar{F} = 8\,t^2\cdot t^2\cdot t^3\ \bar{i} + 5\,t^3\,\bar{j} - 4\cdot t\cdot t^2$

$= 8\,t^7\ \bar{i} + 5\,t^3\,\bar{j} - 4\,t^3\,\bar{k}$

$\bar{F}\cdot d\bar{r} = [8\,t^7 + (2t)(5t^3) + 3t^2(-4t^3)]\,dt$

$= (8t^7 + 10t^4 - 12t^5)\,dt$

$\int_c \bar{F}\cdot d\bar{r} = \int_0^1 [8\,t^7 + 10\,t^4 - 12\,t^5]\,dt$

$= [t^8 + 2t^5 - 2t^6]_0^1$

$= [1 + 2 - 2] - 0 = 1$

Ex. Evaluate $\int_c \bar{F}\cdot d\bar{r}$ for the vector valued field $\bar{F} = y\bar{i} - x\bar{j}$ conterclockwise direction along the unit circle $x^2+y^2=1$, from $(1,0)$ & $(0,1)$

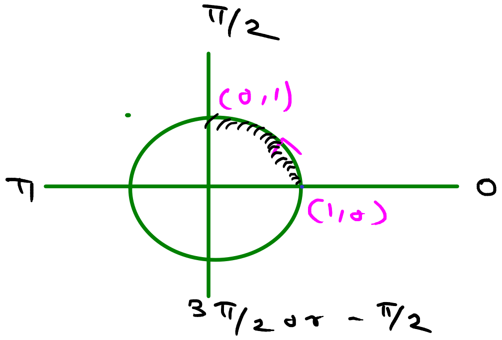
$\Rightarrow$ $x = 1\cos t,\ y = 1\sin t$, $0 \le t \le \frac{\pi}{2}$

$x^2+y^2=a^2$
Anti-clock
$x = a\cos t,\ y = a\sin t$
clock
$x = a\cos t$
$y = -a\sin t$

$\bar{r} = x\bar{i} + y\bar{j}$

$\bar{r} = \cos t\,\bar{i} + \sin t\,\bar{j}$

$d\bar{r} = -\sin t\,dt\ i + \cos t\,dt\,\bar{j}$



$\bar{F} = \sin t\,\bar{i} - \cos t\,\bar{j}$

$\bar{F}\cdot d\bar{r} = [-\sin^2 t - \cos^2 t]\,dt$

$= -dt$

$\therefore \int_c \bar{F}\cdot d\bar{r} = \int_0^{\pi/2} -dt = -[t]_0^{\pi/2} = -[\frac{\pi}{2} - 0] = -\frac{\pi}{2}$

Ex. Evaluate $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = xz\bar{i} - yz\bar{j}$ and c is the line segment from $(-1,2,0)$ to $(3,0,1)$

$\Rightarrow \quad \bar{r}(t) = (1-t)(-1,2,0) + t(3,0,1), \quad 0 \le t \le 1$

$= (-1+t, 2-2t, 0) + (3t, 0, t)$

$= (-1+4t, 2-2t, t)$

$\bar{r}(t) = (-1+4t)\bar{i} + (2-2t)\bar{j} + t\bar{k}$

$d\bar{r} = 4\,dt\,\bar{i} - 2dt\,\bar{j} + dt\,\bar{k}$

$x = 4t-1, \quad y = 2-2t, \quad z = t$

$\bar{F} = (4t-1)t\,\bar{i} - (2-2t)t\,\bar{j}$

$= (4t^2-t)\bar{i} - (2t-2t^2)\bar{j} + 0\bar{k}$

$\bar{F}\cdot d\bar{r} = [(4t^2-t)4 + (2t-2t^2)2 + 0]\,dt$

$= [16t^2 - 4t + 4t - 4t^2]\,dt$

$= 12t^2\,dt$

$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 12t^2\,dt = 12\left[\frac{t^3}{3}\right]_0^1 = 4[1-0] = 4$

Ex. Evaluate $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F}(x,y) = y^2\bar{i} + (x^2-4)\bar{j}$

and c is the portion of $y = (x-1)^2$ from $x=0$ to $x=3$

$\Rightarrow$ Take $x = t$

$\Rightarrow y = (t-1)^2, \quad 0 \le t \le 3$

$\bar{r}(t) = t\bar{i} + (t-1)^2\bar{j}$

$d\bar{r} = dt\,\bar{i} + 2(t-1)dt\,\bar{j}$

$\bar{F} = (t-1)^4\,\bar{i} + (t^2-4)\bar{j}$

$$\bar{F}\cdot d\bar{r} = [(t-1)^4 + 2(t-1)(t^2-4)]\,dt$$

$$= ((t-1)^4 + 2[t^3-4t-t^2+4])\,dt$$

$$= [(t-1)^4 + 2t^3 - 8t - 2t^2 + 8]\,dt$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 [(t-1)^4 + 2t^3 - 8t - 2t^2 + 8]\,dt$$

$$= \left[\frac{(t-1)^5}{5} + \frac{2t^4}{4} - 8\frac{t^2}{2} - 2\frac{t^3}{3} + 8t\right]_0^1$$

$$= \left[0 + \frac{1}{2} - 4 - \frac{2}{3} + 8\right] - \left(-\frac{1}{5}\right)$$

$$= \frac{1}{2} - \frac{2}{3} + 4 + \frac{1}{5}$$

$$= -\frac{1}{6} + \frac{21}{5}$$

$$= \frac{-5+126}{30} = \frac{121}{30}$$

Ex. Evaluate $\int_C \bar{F}\cdot\bar{T}\,ds$ for the vector field $\bar{F} = x^2\bar{i} - y\bar{j}$ along the curve $x = y^2$ from $(4,2)$ to $(1,-1)$

$$\Rightarrow \quad \int_C \bar{F}\cdot\bar{T}\,ds = \int_C \bar{F}\cdot d\bar{r}$$

take $y = t \Rightarrow x = t^2$, $2 \le t \le -1$

$$\bar{r} = t^2\bar{i} + t\bar{j}$$

$$d\bar{r} = 2t\,dt\,\bar{i} + dt\,\bar{j}$$

$$\bar{F} = t^4\bar{i} - t\bar{j}$$

$$\bar{F}\cdot d\bar{r} = [t^4(2t) - t]\,dt$$

$$= (2t^5 - t)\,dt$$

$$\int_C F\cdot dr = \int_2^{-1} (2t^5 - t)\,dt$$

$$= \cancel{2}^{\,1}\,\frac{t^6}{\cancel{6}\,3} - \frac{t^2}{2}\Bigg]_2^{-1}$$

$$= \left[\frac{1}{3} - \frac{1}{2}\right] - \left[\frac{64}{3} - 2\right]$$

$$= \frac{1}{3} - \frac{1}{2} - \frac{64}{3} + 2$$

$$= -\frac{63}{3} + \frac{3}{2}$$

$$= -21 + \frac{3}{2}$$

$$= \frac{-42+3}{2}$$

$$= -\frac{39}{2}$$

Ex. Evaluate $\int_C xy\,dx + (x+y)\,dy$ along the curve $y = x^2$ from $(-1,1)$ to $(2,4)$

$\Rightarrow$ Take $x = t,\ y = t^2,\quad -1 \le t \le 2$

$dx = dt,\ dy = 2t\,dt$

$$\int_C xy\,dx + (x+y)\,dy = \int_{-1}^{2} t\cdot t^2\,dt + (t+t^2)\,2t\,dt$$

$$= \left[\frac{t^4}{4} + 2\frac{t^3}{3} + 2\frac{t^4}{4}\right]_{-1}^{2}$$

$$= \left[\cancel{\frac{16}{4}}\,4 + \frac{16}{3} + \cancel{\frac{32}{4}}\,8\right]$$

$$- \left[\frac{1}{4} - \frac{2}{3} + \frac{2}{4}\right]$$

$$= \left[12 + \frac{16}{3} - \frac{3}{4} - \frac{2}{3}\right]$$

$$= \frac{45}{4} + \frac{14}{3}$$

$$= \frac{135+56}{12} = \frac{191}{12}$$

Ex. Evaluate $\int_C (x-y)\,dx$, where C is $x=t$,

$y = 2t+1$, $0 \le t \le 3$

$\Rightarrow$ $x=t$, $y=2t+1$

$dx = dt$

$$\int_C (x-y)\,dx = \int_0^3 [t-(2t+1)]\,dt$$

$$= \int_0^3 (-t-1)\,dt$$

$$= -\frac{t^2}{2} - t\Big]_0^3$$

$$= \left(-\frac{9}{2}-3\right) - 0$$

$$= -\frac{15}{2}$$

Ex. Evaluate the line integral

$\int_C -y\,dx + z\,dy + 2x\,dz$ when C is the helix

$\bar{r}(t) = \cos t\,\bar{i} + \sin t\,\bar{j} + t\,\bar{k}$, $0 \le t \le 2\pi$

$\Rightarrow$ $x = \cos t$, $y = \sin t$, $z = t$

$dx = -\sin t\,dt$, $dy = \cos t\,dt$, $dz = dt$

$$\int_C -y\,dx + z\,dy + 2x\,dz$$

$$= \int_0^{2\pi} -\sin t(-\sin t\,dt) + t(\cos t\,dt) + 2\cos t\,dt$$

$$= \int_0^{2\pi} [\sin^2 t + t\cos t + 2\cos t]\,dt$$

$$= \int_0^{2\pi} \left[\frac{1}{2}(1-\cos 2t) + t\cos t + 2\cos t\right] dt$$

$$= \left[\frac{1}{2}t - \frac{1}{2}\frac{\sin 2t}{2} + t(\sin t) + \cos t + 2\sin t\right]_0^{2\pi}$$

$$= \left[\frac{2\pi}{2} - \frac{1}{4}\sin 4\pi + 2\pi\sin 2\pi + \cos 2\pi + 2\sin 2\pi\right]$$
$$- [0 - 0 + 0 + \cos 0 + 0]$$

$= (\pi - 0 + 0 + 1 + 0) - 1$

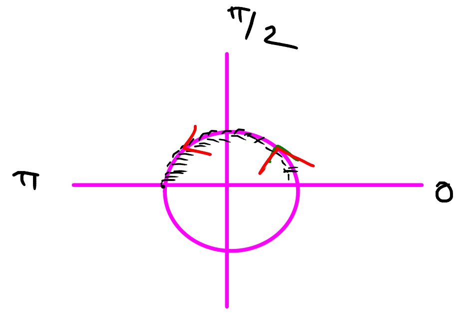
$= \pi + 1 - 1 = \pi$

Ex. Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F}(x,y) = 3\bar{i} + (xy - 2x)\bar{j}$ for each of the following curves.

a) C is the upper half of the circle centered at the origin of radius 4 with counter clockwise rotation.

b) C is the upper half of the circle centered at the origin of radius 4 with clockwise direction.

$\Rightarrow$ a) $x = 4\cos t,\ y = 4\sin t,\quad 0 \le t \le \pi$



$\bar{r}(t) = 4\cos t\,\bar{i} + 4\sin t\,\bar{j}$

$d\bar{r} = -4\sin t\,dt\,\bar{i} + 4\cos t\,dt\,\bar{j}$

$\bar{F} = 3\bar{i} + (16\cos t\sin t - 8\cos t)\bar{j}$

$\bar{F} \cdot d\bar{r} = [-12\sin t + 64\cos^2 t\sin t - 32\cos^2 t]\,dt$

$\int_C \bar{F} \cdot d\bar{r} = \int_0^\pi -12\sin t\,dt + 64\int_0^\pi \cos^2 t\sin t\,dt - 32\int_0^\pi \cos^2 t\,dt$

put $\cos t = m$

$-\sin t\,dt = dm$

$\sin t\,dt = -dm$

$t = \pi \Rightarrow m = -1,\ t = 0 \Rightarrow m = 1$

$= 12[\cos t]_0^\pi + 64\int_1^{-1} m^2(-dm) - 32\int_0^\pi \frac{1+\cos 2t}{2}\,dt$

$= 12[\cos\pi - \cos 0] - 64\left[\frac{m^3}{3}\right]_1^{-1} - 16\left[t + \frac{\sin 2t}{2}\right]_0^\pi$

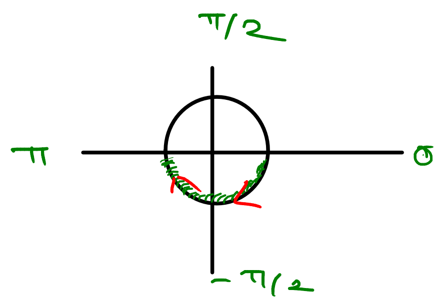
$= 12(-1-1) - \frac{64}{3}(-1-1) - 16(\pi + 0)$

$= -24 + \frac{128}{3} - 16\pi$

$= \frac{-72 + 128}{3} - 16\pi$

$= \frac{56}{3} - 16\pi$

(b) $x = 4\cos t,\ y = -4\sin t,\quad 0 \le t \le \pi$



$$\bar{r}(t) = 4\cos t\, \bar{i} - 4\sin t\, \bar{j}$$

$$d\bar{r} = -4\sin t\, dt\, \bar{i} - 4\cos t\, dt\, \bar{j}$$

$$\bar{F} = 3\bar{i} + (-16\cos t \sin t - 8\cos t)\bar{j}$$

$$\bar{F}\cdot d\bar{r} = [12\sin t + 64\cos^2 t \sin t + 32\cos^2 t]\, dt$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^{\pi} -12\sin t\, dt + 64\int_0^{\pi} \cos^2 t \sin t\, dt + 32\int_0^{\pi} \cos^2 t\, dt$$

$$= -12[\cos t]_0^{\pi} + 64\int_1^{-1} m^2(-dm)$$

$$+ \frac{32}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{\pi}$$

$$= -12[\cos\pi - \cos 0] - 64\left[\frac{m^3}{3}\right]_1^{-1}$$

$$+ 16\left[t + \frac{\sin 2t}{2}\right]_0^{\pi}$$

$$= -12(-1-1) - \frac{64}{3}[-1-1] + 16(\pi)$$

$$= 24 + \frac{128}{3} + 16\pi$$

$$= \frac{72 + 128}{3} + 16\pi$$

$$= \frac{200}{3} + 16\pi.$$

Reference: Vector Calculus, text book for S.Y.B.Sc. by Golden series, Nirali Prakashan.